The Bell System Technical Journal

Devoted to the Scientific Engineering Aspects of Electrical Communication

July, 1923

Transient Oscillations in Electric Wave-Filters

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I. Introduction

THE electric wave-filter has been very fully discussed with respect to its remarkable steady-state properties.[1] In the present paper it is proposed to give the results of a fairly extensive theoretical study of its behavior in the transient state. This study is of particular interest and importance in connection with the wave-filter, because, as we shall find, its remarkable selective characteristics are peculiarly properties of the steady state and become sharply defined only as the steady state is approached. To this fact, it may be remarked in passing, is to be ascribed the uniform failure of wave-filters to suppress irregular and transient interference, such as "static," in anything like the degree with which they discriminate against steady-state currents outside the transmission range. This limitation is common to all types of selective networks and restricts the amount of protection it is possible to secure from transient or irregular interference. In fact the general conclusions of the present study are applicable to all types of selective circuits.

In the present paper the discussion will be principally concerned with the following phases of the general problem.

1. *The indicial admittances of a representative set of wave-filters.* The *indicial admittance*, as explained below, is equal to the current, expressed as a time function, in response to a uniform steady e.m.f. of unit value, applied to the network at time $t=0$. It has been shown in previous papers that a knowledge of the indicial admittance of an invariable network completely determines its behavior, both in the transient and steady state; to all types of applied forces. Its determination is therefore fundamental to the whole problem.

2. *The mode in which the steady-state is built up after a sinusoidal voltage within the frequency transmission range is applied to the wave-*

[1] Physical Theory of the Electric Wave-Filter, G. A. Campbell, *B. S. T. J.*, Nov., 1922; Theory and Design of Uniform and Composite Electric Wave-Filters, O. J. Zobel, *B. S. T. J.*, Jan., 1923.

filter. Formulas are deduced and a set of representative curves computed and plotted which show the dependence of the building-up process on the constants and number of sections of the filter and the frequency of the applied e.m.f. The outstanding deduction from this phase of the problem is that as the selectivity of the filter is increased either by narrowing the transmission band or increasing the number of sections, the time required for sinusoidal currents to build up is proportionally increased. This fact, it may be remarked, sets a theoretical limit to the amount of selectivity which can be employed in communication circuits.

3. *The character and duration of the transient current when a sinusoidal voltage outside the frequency transmission range is applied to the filter.* It will be found that in this case a transient disturbance penetrates the filter which is enormous compared to the final steady state. The magnitude of this disturbance decreases very slowly with the number of filter sections and its duration increases therewith. This phenomenon is an important special case of the general limitations of the selectivity of the filter in the transient state.

4. *The energy which penetrates through selective circuits from random interference.* The energy spectrum of random interference, that is, interference from random disturbances is discussed and a formula is deduced which defines the *figure of merit* of a selective circuit with respect to random interference. This formula leads to general deductions of practical importance regarding the relative merits of selective networks in the transient state and their inherent limitations. It also provides a method for experimentally determining the spectrum of random interference.

Unfortunately the complexity of transient phenomena is such as to absolutely require a large amount of mathematical analysis. Consequently, while the mathematics has been relegated as far as possible to Appendices, a considerable amount necessarily appears in the text. The writers, however, have endeavored to emphasize the physical significance of the mathematics and have included only that which is absolutely essential to an understanding of the problem and the appropriate methods of attack.

In order to keep the analysis within manageable limits and in a form to admit of relatively simple and instructive interpretation, the formulas will be restricted for the most part to non-dissipative filters and the effects of terminal reflections will be ignored.[2] These

[2] The general solution for the case of arbitrary terminal impedances is given in Appendix IV and briefly discussed.

restrictions are desirable on their own account, because the selective properties, both in the transient and steady-state, are isolated and exhibited in the clearest manner when the disturbing effects of dissipation and reflections are absent. As regards dissipation, its effect is usually small for filters of ordinary length and, as regards transient phenomena, is always of such a character as to require no essential modification of the conclusions reached from a study of the ideal non-dissipative filter. In fact the conclusions reached in this paper regarding the inherent limitations of selective circuits in the transient state are conservative.

II. General Theory and Formulas

Before taking up the investigation of wave-filters it is necessary to write down the fundamental formulas of electric circuit theory, which are required in the analysis, and briefly discuss their application to the investigation of transient phenomena in networks in general. The theory and calculation of electrical networks may be approached in a number of ways, as for example, from the Fourier integral.[3] Perhaps the simplest way, however, is to base the theory on the fundamental formulas

$$I(t)=\frac{d}{dt}\int_0^t f(t-y)A(y)dy, \qquad \text{I}$$

and

$$1/pZ(p)=\int_0^\infty e^{-pt}A(t)dt. \qquad \text{II}$$

In these formulas $I(t)$ is the current (expressed as an explicit time function) in any branch or mesh of an electric network which flows in response to the electromotive force $f(t)$ which is applied to the network at time $t \geqq 0$ in the same or any other branch or mesh of the network. The function $A(t)$ is a characteristic function of the constants and connections of the network only which may be termed the *indicial admittance* or the *Heaviside Function*. Its physical significance may be inferred by setting $f(t)=1$, whence it follows that $I(t)=A(t)$. That is to say $A(t)$ *is equal to the current in response to a "unit e.m.f." (zero before, unity after time* $t=0$).

In the following we shall be principally concerned with the case when the applied electromotive force is sinusoidal. To deal with this case we set $f(t)=\sin(\omega t+\Theta)$ and equation I becomes

$$I(t)=a(\omega,t)\sin(\omega t+\Theta)+b(\omega,t)\cos(\omega t+\Theta) \qquad \text{III}$$

[3] The Solution of Circuit Problems, T. C. Fry, *Phys. Rev.*, Aug., 1919.

where, denoting $d/dt\, A(t)$ by $A'(t)$,

$$a(\omega, t) = A(o) + \int_0^t \cos \omega y\, A'(y)\, dy$$

and IV

$$b(\omega, t) = -\int_0^t \sin \omega y\, A'(y) dy.$$

The ultimate steady-state amplitudes are evidently the limits of the foregoing as t approaches infinity. Thus if we write the steady-state current as

$$I = \alpha(\omega) \sin (\omega t + \Theta) + \beta(\omega) \cos (\omega t + \Theta),$$

then

$$\alpha(\omega) = A(o) + \int_0^\infty \cos \omega y\, A'(y) dy$$

and V

$$\beta(\omega) = -\int_0^\infty \sin \omega y\, A'(y) dy.$$

For the derivation and a fuller discussion of the foregoing formulas the reader is referred to " Theory of the Transient Oscillations of Electrical Networks and Transmission Systems," Proc. A. I. E. E., March, 1919.

In the majority of the more important selective networks $A(o) = 0$; that is to say the initial value of the current is zero and the current in response to the applied sinusoidal voltage of the frequency $\omega/2\pi$ is built up entirely from the progressive integrals

$$a(\omega, t) = \int_0 \cos \omega y\, A'(y) dy$$

and

$$b(\omega, t) = -\int_0^t \sin \omega y\, A'(y) dy$$

in accordance with formula IV. The derivative $A'(t) = d/dt\, A(t)$ of the indicial admittance which appears in the integrals will be termed the *impulse function* of the network to indicate its direct physical significance; it is equal to the current in response to a "pulse" of infinitesimal duration and moment (or time integral) unity, or, stated in the terminology of the radio engineer, it is equal to the response of the network to "shock-excitation." These formulas therefore establish a definite quantitative relation between the selective properties of the network and its response to "shock-excitation"; a relation which is of great importance in understanding and interpreting the behavior of selective networks to transient disturbances.[4]

The indicial admittance $A(t)$ is calculable from and may be regarded as defined by the very compact formula II.[5] In this equation $Z(p)$ is the operational impedance of the network. It is derived from the differential equations of the problem by replacing the differential operator d/dt by the symbol p, thus formally reducing the equations to an algebraic form from which the ratio $1/Z(p)$ of the current to electromotive force is gotten by ordinary algebraic processes. $Z(p)$ will involve the constants and connections of the network and will depend, of course, on the mesh or branch in which the electromotive force is inserted and that in which the required current is measured.

The procedure in formulating the transient behavior of networks is as follows. Derive the operational impedance $Z(p)$ as stated above. With $Z(p)$ formulated, the corresponding indicial admittance $A(t)$ is determined by the integral equation II. The appropriate methods of solution of the integral equation are briefly discussed in "The Heaviside Operational Calculus." Sometimes the solution can be recognized by inspection as in the case of the low pass wave-filter. Otherwise the procedure in general is to expand $1/Z(p)$ in such a form that the individual terms of the expansion are recognizable as identical with infinite integrals of the required type. Two expansions of this kind lead to the Heaviside Expansion and power series solution, respectively. The appropriate form of expansion depends on the particular problem in hand and often calls for considerable ingenuity and experience. An excellent illustration of the appropriate process is furnished by the detailed derivation [6] of the indicial admittance of the band pass filter which is rather intricate.

In connection with the problem of the energy absorbed from forces of finite duration, and from random interference, the following formulas are required, of which VIII and IX are original and hitherto unpublished. Formula X, which is a special case of VIII and IX was derived by Rayleigh (*Phil. Mag.*, Vol. 27, 1889, p. 466), in connection with an investigation of the spectrum of complete radiation.

If an applied force $f(t)$ exists only in the finite time interval $0 \leqq t \leqq T$, during which it has a finite number of discontinuities and a

[4] It may be noted in passing that these formulas show the futility of attempting, as so many inventors have done in connection with the problem of protection from "static" disturbances, to design a circuit, which, in the language of patent specifications, shall be unresponsive to shock excitation while at the same time shall be sharply responsive to sustained forces.

[5] The Heaviside Operational Calculus, J. R. Carson, *B. S. T. J.*, Nov., 1922.

[6] See Appendix I.

finite number of maxima and minima, it is representable by the Fourier integral

$$f(t)=\frac{1}{\pi}\int_0^\infty |F(\omega)| \cos [\omega t+\Theta(\omega)]\, d\omega, \qquad \text{VI}$$

where

$$|F(\omega)|^2=\left[\int_0^T f(t) \cos \omega t\, dt\right]^2+\left[\int_0^T f(t) \sin \omega t\, dt\right]^2. \qquad \text{VII}$$

Let this force be applied to a network in branch 1 and let the resultant current $I_n(t)$ be measured in branch n. Let the steady-state transfer impedance at frequency $\omega/2\pi$ be denoted by $Z_{1n}(i\omega)$ and let $z_n\ (i\omega)$ and $\cos \Theta_n$ denote the impedance and power factor of branch n at frequency $\omega/2\pi$. It may then be shown that

$$W'=\int_0^\infty [I_n(t)]^2 dt=\frac{1}{\pi}\int_0^\infty \frac{|F(\omega)|^2}{|Z_{1n}(i\omega)|^2}\, d\omega \qquad \text{VIII}$$

and, as special cases,

$$\int_0^\infty [A'_{1n}(t)]^2 dt=\frac{1}{\pi}\int_0^\infty \frac{d\omega}{|Z_{1n}\,(i\omega)|^2}, \qquad \text{IX}$$

and

$$\int_0^T [f(t)]^2 dt=\frac{1}{\pi}\int_0^\infty |F(\omega)|^2 d\omega. \qquad \text{X}$$

The total energy W, absorbed by branch n from the applied force is given by

$$W=\frac{1}{\pi}\int_0^\infty \frac{|F(\omega)|^2}{|Z_{1n}\,(i\omega)|^2}\, |z_n\,(i\omega)| \cos \Theta_n \cdot d\omega. \qquad \text{VIIIa}$$

Comparison of the formulas for W′ and W shows that, if the branch n is a simple series combination of impedance elements, W′ *is the energy absorbed by a unit resistance element in branch n from the applied force f(t).*

In the subsequent discussion of the behavior of selective circuits to random interference and applied forces of finite duration, W′ of formula VIII will be taken, therefore, as a measure of the energy absorbed by the receiving branch or element. Similarly formula IX measures the energy absorbed when the applied force is impulsive. The application of formula VIII rather than VIIIa, when they differ except for a constant, is justified because we are concerned with the energy absorbed by a receiving element proper, which can be represented by a simple resistance.

The advantage of formula VIII, in addition to its simplicity, resides in the fact that the right hand side is usually quite easily computed, since the integrand is everywhere positive, and this without any explicit reference to the transient phenomena themselves. Formula IX is of particular importance, because, as will be shown in a subsequent part of this paper, it represents, except in limiting cases, the relative amount of energy absorbed from random interference.

III. The Indicial Admittances of Wave-Filters

We are now in possession of the necessary formulas and mathematical processes for investigating the behavior of wave-filters in the transient state. We shall first write down the indicial admittances of the representative types investigated, their derivation being discussed in Appendix I. The formulas given for the low pass and the high pass are exact, while those of the band pass filters are approximations based on the assumption that the transmission band-width is small compared with the "mid-frequency" of the transmission band. They are therefore formally restricted in their application to "narrow band" filters. The analysis of the exact formula, given in Appendix I, shows, however, that the deductions drawn from the approximate formulas of the text are quite generally applicable without errors of any practical consequence to band pass wave-filters, even when the transmission band is relatively wide. These questions are fully discussed in the Appendix.

In the formulas given below the filters are assumed to be infinitely long and the voltage to be applied at "mid-series" position to the initial or zero-th section. $A_n(t)$ is then equal to the current in the nth section in response to a unit voltage (zero before, unity after time $t=0$.)

1. *Low Pass Wave-Filter, Type L_1C_2, Fig.* 1.

$$A_n(t)=\frac{1}{k}\int_0^x J_{2n}(x)dx, \tag{1a}$$

where $x=\omega_c t$,

$\omega_c=2/\sqrt{L_1C_2}=2\pi$ times the critical or cut-off frequency,

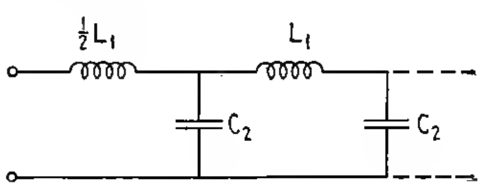

Fig. 1

$J_{2n}(x)$ = The Bessel function of the order $2n$ and argument x, and the filter elements, in terms of the parameters w_c and k, are given by

$$\omega_c = 2/\sqrt{L_1 C_2}, \qquad L_1 = 2k/\omega_c,$$

$$k = \sqrt{L_1/C_2}, \qquad C_2 = 2/\omega_c k.$$

For values of time such that $x < 2n$, $A_n(t)$ is very small and positive, while for $x > 2n$, the character of the solution is exhibited by the approximate solution

$$A_n(t) = \frac{1}{k}\left[1 + \sqrt{\frac{2}{\pi x}}\,\frac{h_{2n}}{q_{2n}}\sin\left(q_{2n}x - \Theta_{2n}\right)\right]. \tag{1b}$$

The formula is deduced from the approximate formulas given in Appendix II for Bessel functions, and h_{2n}, q_{2n} and Θ_{2n} are determined by

$$h_n = \left(\frac{1}{1-n^2/x^2}\right)^{1/4},$$

$$q_n = \sqrt{1-n^2/x^2},$$

and

$$\Theta_n = \frac{2n+1}{4}\pi - n\sin^{-1}(n/x).$$

For sufficiently large values of x, $A_n(t)$ is ultimately given by the asymptotic formula

$$A_n(t) \sim \frac{1}{k}\left[1 + \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{2n+1}{4}\pi\right)\right]. \tag{1c}$$

Formula (1a) was first given by one of the writers (Trans. A. I. E. E., 1919) as a special case of the solution for the dissipative low pass filter (series resistance and shunt leakage).

2. *High Pass Wave-Filter, Type C_1L_2, Fig. 2.*

$$A_n(t) = \frac{1}{k}\left\{\phi_0(x) - \frac{2n}{1!}D^{-1}\phi_1(x) + \frac{(2n)(2n-1)}{2!}D^{-2}\phi_2(x) - \ldots \right.$$

$$\left. \ldots + D^{-2n}\phi_{2n}(x)\right\} \tag{2a}$$

where

$x_c = \omega_c t$,

$\omega_c = 2\pi$ times critical frequency *below* which the filter attenuates,

$C_1 = 1/2\omega_c k$, $\quad k = \sqrt{L_2/C_1}$,

$L_2 = k/2\omega_c$, $\quad \omega_c = 1/2\sqrt{L_2 C_1}$.

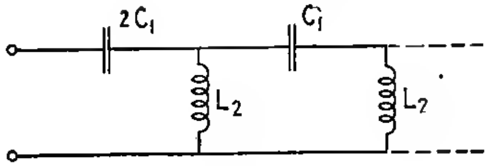

Fig. 2

The symbol D^{-m} denotes multiple integrations, repeated m times and

$$\phi_m(x) = J_o(x) - \frac{m}{1!}J_1(x) + \frac{(m)\ (m-1)}{2!}J_2(x) + \ldots + (-1)^m J_m(x).$$

A large amount of time and effort have been devoted to an attempt to reduce this and other forms of solution (see Appendix I) to a form in which its properties would be exhibited by direct inspection, but without success. Numerical computations and curves must, therefore, be largely relied upon in the study of the high pass filter in the transient state.

For sufficiently large values of x $(x > 4n^2)$ the ultimate behavior of the filter is shown by the asymptotic formula

$$A_n(t) \sim (-1)^n \frac{1}{k}\sqrt{\frac{2}{\pi x}} \cos(x - \pi/4). \tag{2b}$$

Band Pass Wave-Filters.

In all the band pass types of filters discussed below the transmission band lies in the frequency range between $\omega_1/2\pi$ and $\omega_2/2\pi$ so that the band width is $(\omega_2 - \omega_1)/2\pi$. We shall write $\sqrt{\omega_1\omega_2} = \omega_m$ and $\omega_2 - \omega_1 = w$. For each type the filter elements are determined by the parameters ω_m, w and a third parameter [7] k which may be so chosen as to fix the magnitude of the impedance of the filter.

[7] The parameter k is equal to the characteristic impedance, both mid-series and mid-shunt, at mid-frequency of the confluent band, "constant k" type of wave-filter. See Theory and Design of Uniform and Composite Electric Wave-Filters, this Journal, Jan., 1923.

The formulas for the indicial admittances of all the band pass filters are approximate, as stated above, and are deduced on the assumption that the band width is narrow. Practically, however, as regards the essential deductions drawn from them, they are not so restricted but are applicable to the case of relatively wide bands. (See Appendix I.)

There are, of course, an infinite variety of band pass filters; the ones investigated in the present paper are, however, representative and the conclusions drawn from a study of them are, in their general aspects, applicable to all types.

3. *Band Pass Wave-Filter, Type $L_1C_1L_2C_2$, Fig.* 3.

$$A_n(t)=\frac{w}{\omega_m k}J_{2n}(y)\sin x \tag{3a}$$

where $x=\omega_m t$; $y=wt/2$; and the filter elements are given by

$$L_1=2k/w, \qquad L_2=wk/2\omega_m^2,$$

$$C_1=w/2k\omega_m^2, \qquad C_2=2/wk.$$

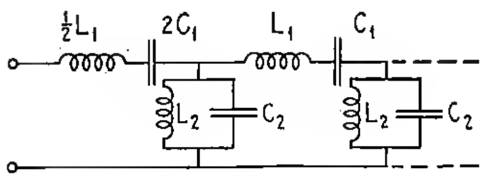

Fig. 3

This is the "constant k" type of filter and, as will be noted, the elements are so proportioned that $L_1C_1=L_2C_2=1/\omega_m^2$, and $L_1/C_2=L_2/C_1=k^2$.

From the properties of Bessel functions discussed in Appendix II, it follows that $A_n(t)$ is very small until $y\geqq 2n$. For values of $y>2n$, the character of the function is clearly exhibited by the following approximate formulas, although these are not sufficiently accurate for the purposes of precise computation.

$$A_n(t)\doteq\frac{w}{\omega_m k}h_{2n}\sqrt{\frac{2}{\pi y}}\cos(q_{2n}y-\Theta_{2n})\sin x \tag{3b}$$

$$\doteq\frac{w}{2\omega_m k}h_{2n}\sqrt{\frac{2}{\pi y}}\left[\sin(x-q_{2n}y+\Theta_{2n})+\sin(x+q_{2n}y-\Theta_{2n})\right] \tag{3c}$$

and ultimately,

$$A_n(t) \sim \frac{w}{2\omega_m k}\sqrt{\frac{2}{\pi y}}\left[\sin\left(x-y+\frac{4n+1}{4}\pi\right)+\sin\left(x+y-\frac{4n+1}{4}\pi\right)\right]. \quad (3d)$$

h_{2n}, q_{2n}, Θ_{2n} are determined by the formulas given in Appendix II,—

$$h_n=\left(\frac{1}{1-n^2/y^2}\right)^{1/4},$$

$$q_n=\sqrt{1-n^2/y^2},$$

and

$$\Theta_n=\frac{2n+1}{4}\pi-n\sin^{-1}(n/y).$$

4. *Band Pass Wave-Filter, Type $L_1C_1C_2$, Fig. 4.*

$$A_n(t)=\frac{w}{\omega_m k}J_n(y)\sin(x-n\pi/2) \quad (4a)$$

where, as above, $x=\omega_m t$; $y=wt/2$, and the filter elements are given by

$$L_1=2k/w;\ C_1=w/2k\omega_1^2;\ C_2=\frac{2}{(\omega_1+\omega_2)k}.$$

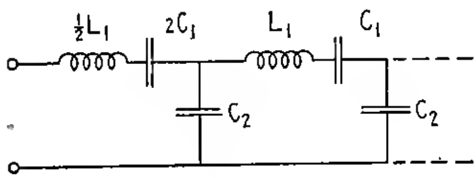

Fig. 4

The approximate formulas for $y>n$, are,

$$A_n(t) \doteq \frac{w}{\omega_m k}h_n\sqrt{\frac{2}{\pi y}}\cos(q_n y-\Theta_n)\sin(x-n\pi/2) \quad (4b)$$

$$\doteq \frac{w}{2\omega_m k}h_n\sqrt{\frac{2}{\pi y}}[\sin(x-q_n y+\Theta_n-n\pi/2)+\sin(x+q_n y-\Theta_n-n\pi/2)] \quad (4c)$$

and ultimately

$$A_n(t) \sim \frac{w}{2\omega_m k}\sqrt{\frac{2}{\pi y}}\left[\sin(x-y+\pi/4)+\sin\left(x+y-\frac{4n+1}{4}\pi\right)\right]. \quad (4d)$$

5. *Band Pass Wave-Filter, Type* $L_1C_1L_2$, Fig. 5.

$$A_n(t) = \frac{w}{\omega_m k} J_n(y) \sin (x+n\pi/2) \tag{5a}$$

where $x=\omega_m t$, $y=wt/2$ and the filter elements are determined by

$$L_1=2\omega_1 k/w\omega_2; \quad C_1=w/2k\omega_m^2; \; L_2=\frac{\omega_1+\omega_2}{2\omega_m^2}k.$$

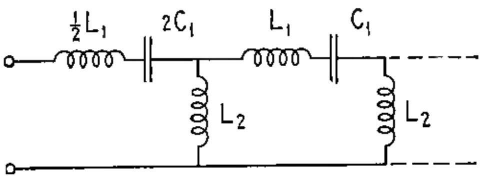

Fig. 5

The approximate formulas for $y>n$ are

$$A_n(t) \doteq \frac{w}{\omega_m k} h_n \sqrt{\frac{2}{\pi y}} \cos (q_n y-\Theta_n) \sin (x+n\pi/2) \tag{5b}$$

$$\doteq \frac{w}{2\omega_m k} h_n \sqrt{\frac{2}{\pi y}} [\sin(x-q_n y+\Theta_n+n\pi/2)+\sin(x+q_n y-\Theta_n+n\pi/2)] \tag{5c}$$

and ultimately

$$A_n(t) \backsim \frac{w}{2\omega_m k} \sqrt{\frac{2}{\pi y}} \left[\sin \left(x-y+\frac{4n+1}{4}\pi\right)+\sin (x+y-\pi/4)\right]. \tag{5d}$$

6. *Band Pass Wave-Filter, Type* $L_1L_2C_2$, *Fig.* 6.

$$A_n(t) = \frac{2w}{\omega_m k} [J_n(y) \sin (x-n\pi/2) - J_n'(y) \cos (x-n\pi/2)] \tag{6a}$$

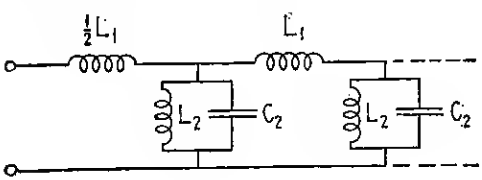

Fig. 6

where $x=\omega_m t$; $y=wt/2$; $J_n'(y)=d/dy\, J_n(y)$, and

$$L_1=\frac{2k}{\omega_1+\omega_2}; \quad L_2=wk/2\omega_1^2; \quad C_2=2/wk.$$

The approximate formulas for $y>n$ are

$$A_n(t) \doteq \frac{2w}{\omega_m k} h_n \sqrt{\frac{2}{\pi y}} [\cos (q_n y - \Theta_n) \sin (x - n\pi/2) + q_n \sin (q_n y - \Theta_n) \cos (x - n\pi/2)] \tag{6b}$$

$$\doteq \frac{w}{\omega_m k} h_n \sqrt{\frac{2}{\pi y}} \begin{bmatrix} (1-q_n) \sin (x - q_n y + \Theta_n - n\pi/2) \\ +(1+q_n) \sin (x + q_n y - \Theta_n - n\pi/2) \end{bmatrix} \tag{6c}$$

and ultimately

$$A_n(t) \sim \frac{2w}{\omega_m k} \sqrt{\frac{2}{\pi y}} \sin\left(x + y - \frac{4n+1}{4}\pi\right). \tag{6d}$$

7. *Band Pass Wave-Filter, Type $C_1L_2C_2$, Fig. 7.*

$$A_n(t) = \frac{2}{\omega_m k}\left(\frac{w}{2\omega_m}\right)^n P + \frac{2w}{\omega_m k} [J_n(y) \sin (x + n\pi/2) + J_n'(y) \cos (x + n\pi/2)], \tag{7a}$$

where $x = w_m t$; $y = wt/2$, and

$$C_1 = \frac{\omega_1 + \omega_2}{2k\omega_m^2};\ L_2 = wk/2\omega_m^2;\ C_2 = 2\omega_1/w\omega_2 k.$$

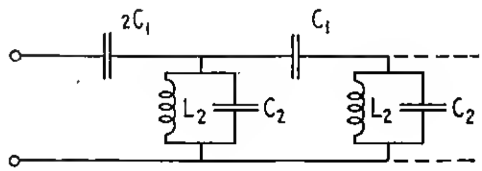

Fig. 7

The symbol P in the first term denotes a "pulse" at time $t=0$; that is

$$P = \infty \text{ at } t=0,$$
$$=0 \text{ for } t>0,$$

and

$$\int_0^\infty P dt = 1.$$

The first term in $A_n(t)$ exists in consequence of the fact that at the instant the voltage is applied the filter behaves like a pure capacity network. For narrow band filters the factor $(w/2\omega_m)^n$ is small so that this term does not contribute appreciably to the steady state. As a matter of fact in actual filters which necessarily have some series resistance, it does not exist.

The approximate formulas for $y>n$ are

$$A_n(t) \doteq \frac{2w}{\omega_m k} h_n \sqrt{\frac{2}{\pi y}} \,[\cos(q_n y-\Theta_n)\sin(x+n\pi/2) - q_n \sin(q_n y-\Theta_n)\cos(x+n\pi/2)] \tag{7b}$$

$$\doteq \frac{w}{\omega_m k} h_n \sqrt{\frac{2}{\pi y}}\,[(1+q_n)\sin(x-q_n y+\Theta_n+n\pi/2) + (1-q_n)\sin(x+q_n y-\Theta_n+n\pi/2)] \tag{7c}$$

and ultimately

$$A_n(t) \sim \frac{2w}{\omega_m k}\sqrt{\frac{2}{\pi y}}\sin\left(x-y+\frac{4n+1}{4}\pi\right). \tag{7d}$$

8. *Discussion of Indicial Admittances.*

The indicial admittances for the low pass filter, that is, the current in response to a steady unit e.m.f. applied at time $t=0$, are shown in the curves of Figs. 8, 9 and 10, for the initial or zero-th, the 3rd and the 5th sections. These curves together with the exact and approximate formulas given above are sufficient to give a reasonably comprehensive idea of the general character of these oscillations and their dependence on the number of sections and the constants of the filter.

It will be observed that the current is small until a time approximately equal to $2n/\omega_c = n\sqrt{L_1C_2}$ has elapsed after the voltage is applied. Consequently the low pass filter behaves as though currents were transmitted with a finite velocity of propagation $\omega_c/2 = 1/\sqrt{L_1C_2}$ sections per second. This velocity is, however, only apparent or virtual since in every section the currents are actually finite for all values of time>0.

After time $t=n\sqrt{L_1C_2}$ has elapsed the current oscillates about the value $1/k$ with increasing frequency and diminishing amplitude. The amplitude of these oscillations is approximately

$$\frac{1/k}{\sqrt{1-(2n/\omega_c t)^2}}\sqrt{\frac{2}{\pi\omega_c t}}$$

and their instantaneous frequency (measured by intervals between zeros)

$$\frac{\omega_c}{2\pi}\sqrt{1-(2n/\omega_c t)^2}.$$

The oscillations are therefore ultimately of cut-off or critical frequency $w_c/2\pi$ in all sections, but this frequency is approached more and more slowly as the number of filter sections is increased.[8]

The indicial admittances of the band pass filter, type $L_1C_1L_2C_2$, are shown in Figs. 11, 12 and 13 for the initial, the 3rd and the 5th sections. These curves show not the actual oscillations but their *envelopes*. That is to say the curves must be multiplied by sin $\omega_m t$ to give the actual oscillations. The "mid-frequency" $\omega_m/2\pi$ may therefore be regarded as the "carrier frequency" which is modulated by the relatively low frequency oscillations shown in the curves.

Comparison of the formulas for the indicial admittances of the band filters of type $L_1C_1C_2$ and $L_1C_1L_2$ with that of type $L_1C_1L_2C_2$ shows that these curves are applicable to the two former types provided the number of sections is doubled and the phase of the oscillations of frequency $\omega_m/2\pi$ is correctly modified.

Referring to Figs. 11, 12, 13 it will be observed that the oscillations are small until time $t=4n/w$; consequently they are transmitted with an apparent velocity of propagation roughly equal to $w/4=1/2\sqrt{L_1C_2}$[9] sections per second.

After time $t=4n/w$, the low frequency oscillations shown in the curves are of increasing frequency and diminishing amplitude, their envelope being roughly equal to

$$\frac{w}{\omega_m k}\sqrt{\frac{4}{\pi w t}}.$$

The actual oscillations are analyzable into two frequencies

$$\frac{1}{2\pi}\left(\omega_m+\frac{w}{2}\sqrt{1-(4n/wt)^2}\right) \text{ and } \frac{1}{2\pi}\left(\omega_m-\frac{w}{2}\sqrt{1-(4n/wt)^2}\right)$$

so that the ultimate oscillations are of the two critical frequencies

$$\frac{1}{2\pi}(\omega_m+w/2) \text{ and } \frac{1}{2\pi}(\omega_m-w/2).$$

[8] For curves showing the indicial admittance of the low pass filter when n is very large, the reader is referred to Transient Oscillations, Trans. A. I. E. E., 1919.

[9] For types $L_1C_1C_2$ and $L_1C_1L_2$ the velocity in sections per second is double this. This corresponds to the fact that two sections of these types are approximately equivalent, as regards their selectivity, to one section of type $L_1C_1L_2C_2$.

For both the low pass and band pass filters the oscillations of the indicial admittances are of continuously variable frequency which traverses the frequency transmission band and ultimately reaches the critical frequencies of the filter.[10]

The indicial admittances of the band pass filter, type $L_1L_2C_2$ are shown in Figs. 14, 15, 16 for the initial, the 6th and 10th sections.[11] The curves show the oscillation envelopes $\sqrt{(J_n^2+J'^2_n)}$, whereas the actual oscillations are within a constant,

$$\sqrt{[J_n^2(wt/2)+J'^2_n(wt/2)]}\ \sin(\omega_m t-n\pi/2-\Theta_n),$$

where $\Theta_n=\tan^{-1}(J_n'/J_n)$. For a narrow band filter the variation in the phase angle Θ_n is very slow.

The principal difference between these curves and the corresponding curves for type $L_1C_1L_2C_2$ is that the envelope of the oscillation does not go through zero as in the latter. In addition the oscillations are ultimately of a single frequency $\frac{1}{2\pi}(\omega_m+w/2)$ while for type $C_1L_2C_2$ the ultimate frequency is $\frac{1}{2\pi}(\omega_m-w/2)$.

The indicial admittance of the high pass filter, shown in the curves of Figs. 17, 18, 19, 20 for the initial, the 1st, 2nd and 3rd sections, differs in important respects from those of the low pass and band pass filters. In the first place the current jumps instantaneously to its maximum value $1/k$ in all sections, so that the velocity of propagation is infinite.[12] After this initial jump the current oscillates with decreasing frequency and decreasing amplitude, the oscillation frequency becoming ultimately the critical or cut-off frequency $\omega_c/2\pi$. The initial frequency and the time required for the oscillation frequency to reduce to $\omega_c/2\pi$, increases, practically linearly with the number of sections. *The oscillation frequency varies continuously and traverses the frequency transmission range of the filter from infinite frequency (represented by the initial jump) down to the critical frequency of the filter, below which it attenuates sinusoidal currents.*

[10] From a purely mathematical viewpoint, this fact explains the transmission, without attenuation, of a continuous band of frequencies.

[11] These curves are applicable to the $C_1L_2C_2$ type of band pass filter, due regard being had to difference in phase, and to the initial jump of current. See formulas (6a) and (7a).

[12] This is, of course, a consequence of the assumption of zero series inductance and shunt capacity. Actually, of course, the circuit must include a finite amount of both.

IV. The Building-Up of Alternating Currents in Wave-Filters

If an e.m.f. $\sin(\omega t+\Theta)$ is applied to the low pass wave-filter (type L_1C_2) at time $t=0$, then by formulas I and (1a), the resultant current in the nth section builds up in accordance with the expression

$$\frac{1}{k}\left[\sin\Theta\int_0^x J_{2n}(x_1)\cos\lambda(x-x_1)dx_1+\cos\Theta\int_0^x J_{2n}(x_1)\sin\lambda(x-x_1)dx_1\right],$$

where $x=\omega_c t$ and $\lambda=\omega/\omega_c$.

For the band pass filter, type $L_1C_1L_2C_2$, the corresponding formula, based on the approximations discussed in the preceding, is by I and (3a).

$$\frac{1}{k}\left[\sin(\mu y+\Theta)\int_0^y J_{2n}(y_1)\cos(\lambda-\mu)(y-y_1)dy_1 \right.$$
$$\left. +\cos(\mu y+\Theta)\int^y J_{2n}(y_1)\sin(\lambda-\mu)(y-y_1)dy_1\right]$$

where $y=wt/2$; $\lambda=2\omega/w$; and $\mu=2\omega_m/w$ so that $\mu y=\omega_m t$. Similar formulas are deducible for the other types of band pass filters considered in the preceding section.

Comparison of these formulas shows that, in both the low pass and band pass wave-filters, the genesis and growth of the current in response to an e.m.f. $\sin(\omega t+\Theta)$, applied at time $t=0$, is mathematically determined by definite integrals of the form

$$S_n(z;\nu)=\int_0^z J_n(z_1)\sin\nu(z-z_1)dz_1,$$

and

$$C_n(z;\nu)=\int_0^z J_n(z_1)\cos\nu(z-z_1)dz_1.$$

These integrals [13] have been extensively studied in the course of this investigation; their general properties and the appropriate methods of computation are discussed in Appendix III.

The subsidence of the current, when a sinusoidal e.m.f. is removed, is also determined by the above formulas for the low pass and band pass filters. To show this suppose that prior to the reference time $t=0$, that steady-state currents are flowing in the filter in response

[13] The writers take pleasure in acknowledging their indebtedness to T. H. Gronwall, consulting mathematician, who furnished asymptotic formulas for the computation of these integrals. See Appendix III.

to an e.m.f. sin $(\omega t+\Theta)$, which is removed at time $t=0$. We can represent this condition correctly by regarding the e.m.f. sin $(\omega t+\Theta)$ as continuing, while a negative e.m.f., $-\sin(\omega t+\Theta)$, is applied at time $t=0$. The resultant current for $t\geqq 0$, is then

$$\alpha_n(\omega)\sin(\omega t+\Theta)+\beta_n(\omega)\cos(\omega t+\Theta)$$
$$-\frac{1}{k}\left[\sin\Theta\cdot C_{2n}(x;\lambda)+\cos\Theta\cdot S_{2n}(x;\lambda)\right]$$

for the low pass filter with a corresponding expression for the band pass. $\alpha_n(\omega)$ and $\beta_n(\omega)$ are the real and imaginary parts of the steady state admittances of the filter at frequency $\omega/2\pi$.

Figs. 21–32 exhibit the phenomena attending the building-up of alternating currents in the low pass filter for a sufficient number of representative cases to show the effects of the length of the filter and the applied frequency. For $\omega_c t>25$, the curves represent the *transient distortion*, that is the difference between the final steady state and actual current. For $\omega_c t<24$ the actual current is shown. An important outstanding result which follows from a study of these curves and the formulas of Appendix III may be stated as follows:

The time T required for an alternating current of frequency $\omega/2\pi$ to build up to its proximate steady state in the nth section of a low pass wave-filter is given approximately by the formula

$$T=\frac{2n}{\omega_c}\frac{1}{\sqrt{1-(\omega/\omega_c)^2}}.$$

The first factor $2n/\omega_c$ represents the delay due to the apparent finite velocity of propagation, while the second factor represents the effect of the applied frequency in its relation to the cut-off frequency of the filter.

This formula is a rather rough approximation when the number of sections n is small. Furthermore the time at which the current reaches its *proximate* steady state does not admit of precise definition.[14] Nevertheless the formula is in substantial agreement with the facts as regards the effect of a number of filter sections, cut-off frequency and applied frequency on the phenomena, and is of great practical importance.[15]

[14] Actually the time T corresponds to a singularity in the mathematical formulas. See Appendix III.

[15] This formula has been applied in the design of periodically loaded cable circuits, which are of such length in the Bell System as to make transient phenomena a factor which must be taken into account. The formula is in close agreement with a large amount of experimental evidence.

The *transient distortion*, it is interesting to note, is, as regards frequency, independent of the applied frequency, and ultimately attains the cut-off frequency of the filter. Its envelope is ultimately

$$\frac{1}{k}\frac{\omega/\omega_c}{1-(\omega/\omega_c)^2}\sqrt{\frac{2}{\pi\omega_c t}}$$

when a voltage sin ωt is applied, and

$$\frac{1}{k}\frac{1}{1-(\omega/\omega_c)^2}\sqrt{\frac{2}{\pi\omega_c t}}$$

when a voltage cos ωt is applied.

Figs. 33 and 34 show the form of the current in the 5th section when sinusoidal voltages sin ωt and cos ωt of frequency 25 per cent above the cut-off frequency of the filter are applied. The transient current shown in the curves increases in frequency up to the critical frequency of the filter, the oscillations being ultimately given by

$$\frac{1}{k}\frac{\omega/\omega_c}{(\omega/\omega_c)^2-1}\sqrt{\frac{2}{\pi\omega_c t}}\cos\left(\omega_c t-\frac{2n+1}{4}\pi\right)$$

and

$$\frac{1}{k}\frac{1}{(\omega/\omega_c)^2-1}\sqrt{\frac{2}{\pi\omega_c t}}\sin\left(\omega_c t-\frac{2n+1}{4}\pi\right)$$

corresponding respectively to applied voltages sin ωt and cos ωt. The amplitude of these transient oscillations are enormous compared with the final steady state, and the curves furnish a clear illustration of the fact, stated in a previous part of this paper, that the selective properties of wave-filters are essentially properties of the steady state only.

Figs. 35–41 show the building-up phenomena in the band pass filter, type $L_1C_1L_2C_2$, and are applicable also to types $L_1C_1C_2$ and $L_1C_1L_2$ when proper values are assigned to the constants and parameters.[16] The curves actually show the envelopes of the oscillations which are of slowly variable frequency in the neighborhood of $\omega_m/2\pi$.

A study of these curves and the formulas of Appendix III lead to the following proposition, analogous to that stated above for the low pass filter.

The time T required for an alternating current of frequency $\omega/2\pi$ within the transmission range $w/2\pi$ of a band filter to build up to its

[16] n sections of type $L_1C_1L_2C_2$ are approximately equivalent to $2n$ sections of type $L_1C_1C_2$ or of type $L_1C_1L_2$.

proximate steady state in the nth section is given approximately by the formula

$$T=\frac{4n}{w}\frac{1}{\left[1-4\left(\frac{\omega-\omega_m}{w}\right)^2\right]^{1/2}}$$

for type $L_1C_1L_2C_2$ and one half this amount for the other types of band pass filters discussed in this paper.

These curves show the envelope of the oscillations with fidelity but are not well adapted to exhibit the actual frequencies. These are given by the formula

$$\sqrt{C^2+S^2}\sin\left[\omega_m t+\Theta+\tan^{-1}(S/C)\right]$$

where C and S denote the definite integrals

$$C_{2n}\left(\frac{wt}{2};\ \frac{2(\omega-\omega_m)}{w}\right)\ \text{and}\ S_{2n}\left(\frac{wt}{2};\ \frac{2(\omega-\omega_m)}{w}\right).$$

The envelope is therefore substantially independent of the phase angle Θ of the applied e.m.f. The frequency is ultimately the applied frequency $\omega/2\pi$. The *transient distortion* is analyzable into two frequencies

$$\frac{1}{2\pi}\left(\omega_m+\frac{w}{2}\sqrt{1-(4n/wt)^2}\right)\ \text{and}\ \frac{1}{2\pi}\left(\omega_m-\frac{w}{2}\sqrt{1-(4n/wt)^2}\right),$$

and its envelope is ultimately

$$\frac{1+4\left(\frac{\omega-\omega_m}{w}\right)^2}{1-4\left(\frac{\omega-\omega_m}{w}\right)^2}\sqrt{\frac{4}{\pi wt}}.$$

The building-up of alternating currents in the high pass filter has been investigated only qualitatively owing to the extremely laborious computations required. The process is essentially different from that in the low pass and band pass filters. When an e.m.f. $\sin(\omega t+\Theta)$ is applied the current in all sections jumps instantly to the value

$$\frac{1}{k}\sin(\omega t+\Theta).$$

Therefore the process depends on the applied frequency. If the applied frequency is within the transmission band ($\omega>\omega_c$), the current builds up to its ultimate frequency, the time required being given approximately by the formula

$$T=\frac{2n\omega_c}{\omega^2}\frac{1}{\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}},$$

(by the principle of stationary phase; see footnote 31).

It thus requires an infinite time when the applied frequency is equal to the critical frequency while infinite applied frequencies build up instantly.

When the applied frequency is outside the transmission band, the current *subsides* to its steady value, the time required being proportional to the ratio n/ω_c and decreasing as the applied frequency is decreased.

The fact that the initial value of the current is of the same order of magnitude as that of steady state currents in the transmission range is an outstanding feature of the process and reflects the failure of the selective properties of this type of filter in the transient state.

V. The Energy Absorbed from Transient Applied Forces

In only a relatively few cases is the solution for the transient current, in response to suddenly applied forces, reducible to a manageable form, which admits of interpretation or of computation without prohibitive labor. Fortunately, however, it is usually possible to calculate the energy absorbed by a receiving element in a selective network from suddenly applied forces of finite time duration and such a calculation throws a great deal of light on the general properties of selective circuits in the transient state. The calculation is based on formulas VI to IX of Section II.

A particularly important example is the energy absorbed from the force $\sin(pt+\Theta)$, applied at time $t=0$ and removed at time $t=T$. If the energy is averaged with respect to the phase angle Θ, we get [17]

$$\int_0^\infty [I(t)]^2 dt = \frac{1}{2\pi}\int_0^\infty \frac{d\omega}{|Z(i\omega)|^2}\left\{\frac{1-\cos(\omega-p)T}{(\omega-p)^2}+\frac{1-\cos(\omega+p)T}{(\omega+p)^2}\right\}.$$

If $p/2\pi$ is in the neighborhood of the frequency which the network is designed to select, this becomes approximately

$$\int_0^\infty [I(t)]^2 dt = \frac{1}{2\pi}\int_0^\infty \frac{1-\cos(\omega-p)T}{(\omega-p)^2}\,\frac{d\omega}{|Z(i\omega)|^2}. \qquad (8)$$

In the *steady state* the time integral of the square of the current in response to the e.m.f. $\sin(pt+\Theta)$ during the time interval T is simply $T/2|Z(ip)|^2$. The expression

$$\frac{|Z(ip)|^2}{\pi T}\int_0^\infty \frac{1-\cos(\omega-p)T}{(\omega-p)^2}\,\frac{d\omega}{|Z(i\omega)|^2} \qquad (9)$$

is therefore *the relative amount of energy actually absorbed from the*

[17] Here $Z(i\omega)$ is the steady-state transfer impedance and the integral measures the energy absorbed by a unit resistance in the receiving branch.

force sin $(pt+\theta)$ *acting during the time interval T, to that calculated on the assumption of a steady state in this interval.*

Calculations of these formulas are of particular interest and importance in multiplex carrier telephone and telegraph systems where they furnish a measure of the interference between channels operating at different frequencies.

In order to exhibit clearly the significance of the formulas without detailed computation, consider an ideal selective circuit, for which in the range $\omega_1 \leqq \omega \leqq \omega_2, |Z(i\omega)| = Z_T$ (a constant) and everywhere else $|Z(i\omega)| = Z_s$ (a constant, very large compared with Z_T). Under these assumptions, formulas (8) and (9) become approximately, for the case when $p > \omega_2$,

$$\frac{T}{2Z_s^2} + \frac{1}{2\pi} \frac{\omega_2 - \omega_1}{(p-\omega_2)\ (p-\omega_1)} \frac{1}{Z_T^2} \tag{8a}$$

and

$$\left(1 + \frac{1}{\pi T} \frac{Z_s^2}{Z_T^2} \frac{\omega_2 - \omega_1}{(p-\omega_2)\ (p-\omega_1)}\right). \tag{9a}$$

These formulas admit of some quite interesting deductions which are applicable to band filters in general.

(1) The energy absorbed in excess of that calculated in the steady state basis is

$$\frac{1}{2\pi} \frac{\omega_2 - \omega_1}{(p-\omega_2)\ (p-\omega_1)} \frac{1}{Z_T^2}.$$

This is independent of the duration of the applied force and of the degree to which the filter discriminates against steady state currents outside the frequency range $\omega_1 \leqq \omega \leqq \omega_2$. It is proportional to the *band width* and inversely to the product $(p-\omega_2)\ (p-\omega_1)$. It follows therefore that *no amount of selectivity will appreciably reduce the energy absorbed from a sinusoidal force of finite duration outside the transmission range of the filter, below the value given above.*

(2) The fractional excess of energy absorbed is given by

$$\frac{1}{\pi T} \left(\frac{Z_s}{Z_T}\right)^2 \frac{\omega_2 - \omega_1}{(p-\omega_2)\ (p-\omega_1)}.$$

This decreases with the duration of the applied force but *increases as the square of the selectivity (Z_s/Z_T) of the filter.* Hence for forces of short duration the energy absorbed may be very large compared with that calculated on the steady state basis.

VI. Random Interference

We have hitherto confined attention to the transient phenomena when the form of the applied voltage was explicitly given. In the problem of the behavior of wave-filters and selective circuits in general to such disturbances as "static" in radio transmission and "noise" in wire transmission this is not the case, and the applied force is usually more or less completely *random*. By this it is meant that the interfering disturbance, which may be supposed to originate in a large number of unrelated sources, varies in an irregular, uncontrollable manner, and is characterized statistically by no predominant frequency. Consequently the wave form of the applied force at any particular instant is entirely indeterminate. This fact makes it necessary to treat the problem as a statistical one, and deal with mean values. In the following we shall derive formulas for the mean *energy* absorbed from random interference; and then define and discuss the selective figure of merit of networks with respect to random interference.

The mathematical treatment of the problem will be based on formulas VI to VIII of section II. To apply these formulas to the problem of random disturbances and their effect on selective networks, consider a long interval of time, or epoch, say from 0 to T. During this epoch we suppose that the network is subjected to a large number of individual impressed forces $f_1(t), f_2(t) \ldots f_n(t)$, which are unrelated and vary in intensity and wave form in an irregular, indeterminate manner, and thus constitute what will be called *random interference*. If we write

$$\sum(t) = f_1(t) + f_2(t) + \ldots + f_n(t),$$

then by VI, $\sum(t)$ is representable as a Fourier integral, thus:

$$\sum(t) = \frac{1}{\pi}\int_0^\infty |F(\omega)| \cos[\omega t + \Theta(\omega)]\, d\omega$$

while, in accordance with formula VIII, the energy absorbed by the selective network from this random interference is measured by [18]

$$W' = \frac{1}{\pi}\int_0^\infty \frac{|F(\omega)|^2}{|Z(i\omega)|^2}\, d\omega.$$

[18] It should be clearly understood that $Z(i\omega)$ is the transfer impedance of the receiving with respect to the driving branch of the network, and that W' is the energy absorbed by a unit resistance located in the former.

We now introduce the function $R(\omega)$ which will be termed the *energy spectrum of the random interference*, and which is defined by the equation

$$R(\omega)=\frac{1}{T}|F(\omega)|^2. \tag{10}$$

Dividing both sides by T and writing $W'/T=\epsilon$, formula VIII becomes

$$\epsilon=\frac{1}{\pi}\int_0^\infty \frac{R(\omega)}{|Z(i\omega)|^2}d\omega. \tag{11}$$

Both ϵ and $R(\omega)$ become independent of T provided the epoch is made sufficiently great, and *ϵ measures the mean energy absorbed per unit time from the random interference.* The practical significance of this formula is contained in the statement that *the required function of the selective network, as regards random interference, is to minimize the ratio of ϵ to the signal energy. Consequently this ratio furnishes an index of the merit of the network.*

In order to rigorously evaluate the integral of formula (11) the energy spectrum $R(\omega)$ of the interference must be completely specified over the entire interval of integration. Obviously this information cannot be deduced without imposing some restrictions on the character of the interference, or making some hypothesis regarding the mechanism in which it originates. On the other hand if the forces $f_1(t), f_2(t) \ldots f_n(t)$ are absolutely random in a strict mathematical sense, it would appear that all frequencies are equally probable in the spectrum $R(\omega)$ and that, consequently, the most probable energy distribution is that which makes $R(\omega)$ a constant, independent of ω. This inference, however, has not been theoretically established; indeed, the problem does not appear to admit of satisfactory solution by the calculus of probabilities. Furthermore, deductions based on the assumption that the interference is random in a strict mathematical sense might well be inadequate for the applications contemplated, and the "most probable" spectrum in serious disagreement with the spectrum of the actual interference [19] to which we wish to apply the results of the present study.

Fortunately, in view of these difficulties, a complete specification of $R(\omega)$ is not at all necessary for a practical solution of the problem. This is a consequence of the following facts:

[19] For example, the spectrum of the interference presented to the terminals of the selective network will be modified by the characteristics of the "transducer," over which the disturbances are transmitted. Thus both in radio and wire systems, the greater attenuation suffered in transmission by high frequencies, will reduce the relative intensity of the high frequency part of the spectrum.

(a) In the case of efficient selective networks, the important contributions to the integral (11) are confined to a finite continuous range of ω which includes, but is not greatly in excess of, the range which the network is designed to select.[20] This fact is a consequence of the impedance characteristics of selective networks and of the following properties of the spectrum $R(\omega)$.

(b) $R(\omega)$ is a continuous, finite function of ω which converges to zero at infinity and is everywhere positive. It possesses no sharp maxima or minima,[21] and its variation with respect to ω, where it exists, is slow. These properties of $R(\omega)$ are believed to be evident from physical considerations, and will not be elaborated.

Now referring to formula (11), since the numerator and denominator of the integrand are everywhere positive, it follows that a value ω_m of ω exists, such that

$$\epsilon=\frac{1}{\pi}R(\omega_m)\int_0^\infty \frac{d\omega}{|Z(i\omega)|^2}.$$

Now suppose that the network is designed to select frequencies in the range $\omega_1 \leqq \omega \leqq \omega_2$. Then from the properties of the network and of the spectrum $R(\omega)$ discussed above, it follows that ω_m lies close to, or within, the range $\omega_1 \leqq \omega \leqq \omega_2$. In any case, if the band $\omega_2-\omega_1$ is made so narrow that the curvature of $R(\omega)$ over the interval is negligible, then with negligible error ω_m may be taken as 2π times the "mid-frequency" of the band. That is to say, with negligible error, ω_m may be defined either as $(\omega_1+\omega_2)/2$ or as $\sqrt{\omega_1\omega_2}$.

The foregoing argument may be summarized in the following proposition:

The mean energy ϵ absorbed per unit time from random interference by a selective network designed to select the band of frequencies corresponding to $\omega_1 \leqq \omega \leqq \omega_2$ is measured by the formula

$$\epsilon=\frac{1}{\pi}\rho R(\omega_m), \tag{12}$$

where ρ denotes the infinite integral

$$\rho=\int_0^\infty \frac{d\omega}{|Z(i\omega)|^2}$$

[20] This statement excludes from present consideration networks, which, like the high pass filter, select an infinite band of frequencies. This limitation, however, is of no practical consequence, because such networks are quite useless as regards random interference. This question will be briefly discussed later.

[21] The existence of sharp maxima and minima would indicate the presence of *systematic* interference, which should not be regarded as part of the random interference.

and $R(\omega_m)$ *is the spectral energy level of the interference at frequency* $\omega_m/2\pi$. ω_m *lies close to or within the band* $\omega_1 < \omega < \omega_2$, *and when this band is sufficiently small with respect to the curvature of* $R(\omega)$, $\omega_m/2\pi$ *may be aken as the mid-frequency of the band.*

Formula (12) is of very considerable practical and theoretical importance. It furnishes a basis for the experimental determination of the energy spectrum $R(\omega)$, and this determination, for any given epoch, can be made as accurate as desired by employing a band filter which selects a sufficiently narrow band of frequencies. It also leads immediately to the following important proposition.

If a selective network is required to select the band of frequencies corresponding to $\omega_1 \leqq \omega \leqq \omega_2$, *the mean energy absorbed per unit time by the network from random interference is necessarily greater than*

$$\frac{1}{\pi}\int_{\omega_1}^{\omega_2} \frac{R(\omega)}{|Z(i\omega)|^2} d\omega \doteq \frac{1}{\pi} R(\omega_m) \int_{\omega_1}^{\omega_2} \frac{d\omega}{|Z(i\omega)|^2}. \tag{13}$$

This formula, therefore, determines the theoretical limit, beyond which it is not possible to discriminate against random interference.

We are now prepared to introduce a formula which defines *the figure of merit of a selective network with respect to random interference.* This formula gives the signal-to-random-interference energy ratio of the network as compared with the corresponding ratio in an ideal reference circuit (defined below).

Let the network, as above, be designed to select frequencies in the band $\omega_1 \leqq \omega \leqq \omega_2$. Then the energy absorbed per unit time from *steady-state* forces in this frequency range is proportional to

$$\sigma = \frac{1}{\omega_2 - \omega_1} \int_{\omega_1}^{\omega_2} \frac{d\omega}{|Z(i\omega)|^2}.$$

The corresponding mean energy absorbed from random interference is proportional to

$$\rho = \int_0^{\infty} \frac{d\omega}{|Z(i\omega)|^2}$$

when the energy level of the interference is corrected to unity.

The ratio $S = \sigma/\rho$ defines the selective figure of merit of the network with respect to random interference.

Stated in words, *the selective figure of merit of a network with respect to random interference is equal to the statistical signal-to-random-interference energy ratio, divided by the corresponding ratio in an ideal band filter which transmits without loss all frequencies in a "unit" band* ($\omega_2 - \omega_1 = 1$), *and absolutely extinguishes all frequencies outside this band.*

In the foregoing argument, the theoretical limitations have been carefully pointed out and even emphasized. In practical applications, however, it is believed that these limitations are of small or negligible importance, and that the formula for and definition of the selective figure of merit furnish all the information, as regards the behavior of selective circuits to random interference, which we are in a position to make use of. Thus the formula is immediately applicable to the problem of determining the effect of band width, number of sections, dissipation, and terminal reflections on the selectivity of filters with respect to random interference. It furnishes likewise, a means of estimating the comparative merits of the very large number of circuits which have been invented for the purpose of eliminating "static" in radio communication, and leads to general deductions of practical value regarding the inherent limitations imposed on the solution of the "static" problem.

The utility and significance of the foregoing formulas will now be illustrated by application to some representative selective circuits. It is easily shown that, to a good approximation, in the case of the low pass filter (type L_1C_2)

$$S=\frac{1}{\omega_c(1+1/16n^2)},$$

and for the band pass filter (type $L_1C_1L_2C_2$)

$$S=\frac{1}{w(1+1/16n^2)}.$$

In these formulas n denotes the number of filter sections while ω_c is 2π times the cut-off frequency of the low pass filter and w is 2π times the transmission band width of the band filter. In both cases the filters are assumed to be terminated in their characteristic impedances and to be non-dissipative.[22] These formulas show at once the effect of band width and number of sections n on the behavior of wave-filters to random interference, and lead to the following proposition.

In filters designed to select a band of frequencies of width w, the ratio of energy transmitted through the network by the signal and by random interference is inversely proportional to the band width and increased inappreciably when the number of sections is increased beyond two.

As regards the effect of dissipation, a second proposition is deducible.

The effect of introducing dissipation into a network designed to select a single frequency or a band of frequencies is always such as to reduce the ratio of signal energy to that absorbed from random interference.

[22] These approximate formulas are in very good agreement with actual calculations for filters terminated in resistances.

An inference drawn from the study of band filters in the preceding section may be stated as follows:

The selective figure of merit of a wave-filter designed to select a finite band of frequencies is approximately proportional to the minimum time required for sinusoidal currents within the transmission band to build up their approximate steady values, divided by the number of filter sections.

Another circuit of practical interest, which has been proposed as a solution of the "static" problem in radio-communication consists of a series of sharply tuned oscillation circuits, unilaterally coupled through amplifiers.[23] This circuit is designed to receive only a single frequency to which all the individual oscillation circuits are tuned. The figure of merit of this circuit is approximately

$$S=\frac{L}{R}\,\frac{2^{2n-2}[(n-1)!]^2}{(2n-2)!}$$

where n denotes the number of sections, or stages, and L and R are the inductance and resistance of the individual oscillation circuits. The outstanding fact in this formula is the slow rate of increase of S with the number of stages. For example, if the number of stages is increased from 1 to 5, the figure of merit increases only by the factor 3.66, while for a further increase in n the gain is very slow. This gain, furthermore, is accompanied by a serious increase in the "sluggishness" of the circuit; that is, in the particular example cited, by an increase of 5 to 1 in the time required for signals to build up to their steady-state.[24]

The outstanding deduction of practical importance to be drawn from the preceding is that, as regards disturbances which are predominantly random, irregular, or discontinuous, it is useless to employ selective circuits of extremely high selectivity. The gain in signal-to-interference ratio is very small when the selectivity is increased beyond a moderate amount, and is only gotten by making the circuit relatively sluggish and slowly responsive.

The preceding discussion is, for the reasons discussed above, not applicable to selective circuits like the high pass filter, which transmit an infinite band of frequencies. Considerable information, however, regarding the behavior of the high pass filter to random disturbances can be gotten by returning to formula (10) and comparing the energy absorbed by the high pass filter, with that absorbed by a *pure-resistance network*. Reference to formula (10) shows that the

[23] See U. S. Patent No. 1173079 to Alexanderson.

[24] When the number of stages n is fairly large, the selective figure of merit becomes proportional to $\sqrt{n}$ and the building-up time to n.

energy absorbed from random disturbances by a pure resistance network is proportional to

$$\int_0^\infty R(\omega)d\omega.$$

The relative amount of energy absorbed by the high pass filter is *greater than*

$$\int^\infty R(\omega)d\omega.$$

The function $R(\omega)$ represents, as above, the statistical energy spectrum of the interference.

Comparison of these formulas shows at once that, unless the energy of the random interference is largely confined in the range $\omega<\omega_c$, little protection is afforded by the high pass filter.

Appendix I

Derivation of Wave-Filter Indicial Admittances

1. *Low Pass Wave-Filter, Type* L_1C_2.

The derivation of the indicial admittance of this type of filter is given in detail by one of the writers in a previous paper.[25] The method of solution there employed, which is quite generally applicable to periodic structures, consists in writing down the Heaviside Expansion formula for the current in the nth section of a filter of s sections in length $(s>n)$, short circuited at the sth section. The expansion is converted into a definite integral by letting s become infinite and the formula becomes that of the indicial admittance of the nth section of an infinitely long filter. For the non-dissipative filter having mid-series termination, this procedure leads to the formula

$$A_n(t)=\frac{1}{k}\int_0^x dx_1\frac{2}{\pi}\int_0^{\pi/2}\cos(2n\lambda)\cdot\cos(x_1\sin\lambda)d\lambda, \qquad x=\omega_c t,$$

which is identifiable, from known formulas, as

$$A_n(t)=\frac{1}{k}\int_0 J_{2n}(x_1)dx_1. \tag{1.1}$$

A much more direct and flexible method of solution and one which avoids the necessity of setting up the Heaviside expansion formula

[25] Transient Oscillations, Trans. A. I. E. E., 1919. This paper should be consulted for the details of this method.

and then converting into a definite integral, is to employ the integral equation II. If $Z_n(p)$ denote the transfer operational impedance of the nth section of the infinitely long low pass filter, we have

$$\frac{1}{Z_n(p)}=\frac{\omega_c}{k}\frac{1}{\sqrt{p^2+\omega_c^2}}\left(\frac{\sqrt{p^2+\omega_c^2}-p}{\omega_c}\right)^{2n} \tag{1.2}$$

and writing $x=\omega_c t$, $F_n(x)=kA_n(t)$, the integral equation II becomes

$$\int_0^\infty F_n(x)e^{-px}dx=\frac{1}{p\sqrt{p^2+1}}\left(\sqrt{p^2+1}-p\right)^{2n}. \tag{1.3}$$

The solution of this integral equation is known[26]; it is

$$F_n(x)=\int_0^x J_{2n}(x_1)dx_1$$

which agrees with the preceding.

The "mid-series" termination is chosen not only for its importance in practical applications but because in general the indicial admittance has been found to take the simplest form when the voltage is applied at this position. This is not always the case, however. For example in the low pass filter if the e.m.f. is applied, not directly at mid-series but through a terminal inductance $L=L_1/2=k/\omega_c$, the integral equation becomes

$$\int_0^\infty F_n(x)e^{-px}dx=\left(1-p(\sqrt{p^2+1}-p)\right)\frac{1}{p\sqrt{p^2+1}}\left(\sqrt{p^2+1}-p\right)^{2n},$$

whence

$$F_n(x)=\int_0^x J_{2n}(x_1)dx_1-J_{2n+1}(x). \tag{1.4}$$

Unless, however, the terminal impedance is related in some simple manner to the constants of the filter, the resulting formula is necessarily complicated.

2. *High Pass Wave-Filter, Type C_1L_2.*

For this type of filter it can be shown, by the first method discussed above in connection with the low pass filter, that the indicial admittance is expressible as the definite integral

$$A_n(t)=\frac{2}{\pi k}\int_1^\infty \frac{\cos(2n\,\sin^{-1}\frac{1}{\lambda})\sin x\lambda\,d\lambda}{\sqrt{\lambda^2-1}}, \tag{2.1}$$

[26] Nielsen, Cylinderfunktionen, page 186, formula 13.

where $x=\omega_c t$. For the case $n=0$, the solution can be recognized as

$$A_0(t)=\frac{1}{k}J_0(x).$$

To attack the problem by means of the integral equation II, we write down the operational transfer impedance

$$\frac{1}{Z_n(p)}=\frac{1}{k}\,\frac{1}{\sqrt{1+\omega_c^2/p^2}}\left(\sqrt{1+\omega_c^2/p^2}-\omega_c/p\right)^{2n}. \tag{2.2}$$

Writing $\omega_c t=x$, and $A_n(t)=\frac{1}{k}F_n(x)$, and substituting in II gives, as the integral equation of the problem

$$\int_0^\infty F_n(x)e^{-px}dx=\frac{1}{p^{2n}}\,\frac{1}{\sqrt{p^2+1}}\left(\sqrt{p^2+1}-1\right)^{2n}. \tag{2.3}$$

The solution of this equation can be expressed in a number of different forms, depending on the type of expansion of the right hand side which we adopt. One form is as follows:

Expansion of the bracketted expression on the right hand side by the binomial theorem and rearrangement gives

$$\int_0^\infty F_n(x)e^{-px}dx=\frac{1}{\sqrt{p^2+1}}\left[\frac{1}{p^{2n}}+\frac{(2n)(2n-1)}{2!}\,\frac{(p^2+1)}{p^{2n}}+\frac{(2n)\,.\,.\,(2n-3)}{4!}\right.$$
$$\left.\times\frac{(p^2+1)^4}{p^{2n}}+\,.\,.\right]-\left[\frac{2n}{1!}\,\frac{1}{p^{2n}}+\frac{(2n)(2n-1)(2n-2)}{3!}\,\frac{(p^2+1)}{p^{2n}}+\,.\,.\,.\,.\,.\right].$$

Recognizing that

$$\int_0^\infty J_0(x)e^{-px}dx=\frac{1}{\sqrt{p^2+1}},$$

the solution, after rearrangement, becomes the terminating series [27]

$$F_n(x)=kA_n(t)$$
$$=\left(1+\frac{4n^2}{2!}D^{-2}+\frac{4n^2(4n^2-2^2)}{4!}D^{-4}+\frac{4n^2(4n^2-2^2)(4n^2-4^2)}{6!}D^{-6}+\,.\,.\right)J_0(x)$$
$$-2n\left(x+\frac{(4n^2-2^2)}{3!}\,\frac{x^3}{3!}+\frac{(4n^2-2^2)(4n^2-4^2)}{5!}\,\frac{x^5}{5!}+\,.\,.\,.\right), \tag{2.4}$$

where D^{-m} indicates multiple integration, repeated m times. Thus

$$D^{-1}J_0(x)=\int_0^x J_0(x_1)dx_1;\quad D^{-2}J_0(x)=\int_0^x dx_1\int_0^{x_1}J_0(x_2)dx_2;\ \text{etc.}$$

[27] This solution has been derived from the definite integral also.

Another type of expansion, leading to the formula given in the text, is suggested by the known identity

$$\int_0^\infty J_n(x)e^{-px}dx=\frac{1}{\sqrt{p^2+1}}(\sqrt{p^2+1}-p)^n.$$

To introduce this identity, we write the integral equation in the form

$$\int_0^\infty F_n(x)e^{-px}dx=\frac{1}{p^{2n}}\frac{1}{\sqrt{p^2+1}}\Big((\sqrt{p^2+1}-p)+(p-1)\Big)^{2n}$$

and expand the bracketted expression by the binomial theorem. Identification of the individual terms and rearrangement gives the terminating series

$$F_n(x)=\phi_0(x)-\frac{2n}{1!}D^{-1}\phi_1(x)+\frac{(2n)(2n-1)}{2!}D^{-2}\phi_2(x)-\ \ldots\ . \qquad -\frac{2n}{1!}D^{-(2n-1)}\phi_{2n-1}(x)+D^{-2n}\phi_{2n}(x),$$

where $\phi_m(x)$ denotes the terminating series

$$\phi_m(x)=J_0(x)-\frac{m}{1!}J_1(x)+\frac{(m)(m-1)}{2!}J_2(x)+\ \ldots\ +(-1)^mJ_m(x)$$

and as above D^{-m} denotes multiple integration.

It is an easy matter to derive solutions in the form of infinite series, as for example power series and Bessel series. These solutions, however, which have been carefully investigated, have not proved manageable for either computation or interpretation. The solutions given above are also unfortunately, extremely difficult to compute or interpret. For computation, numerical integration of the following difference equations, is sometimes preferable

$$\begin{aligned} &F_0(x)=J_0(x),\\ &F_1(x)-F_0(x)=2\int_0^x dx_1\int_0^{x_1}F_0(x_2)dx_2-2x,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &F_{n+1}(x)-2F_n(x)+F_{n-1}(x)=4\int_0^x dx_1\int_0^{x_1}F_n(x_2)dx_2,\quad n\geqq 1. \end{aligned} \tag{2.5}$$

3. *Band Pass Wave-Filter.*

The mathematical discussion of the band pass filters will be limited to the $L_1C_1L_2C_2$ type shown in Fig. 3. This type is representative and the appropriate mathematical procedure is essentially the same for all the band pass wave-filters.

The first method of solution outlined above for the low pass and high pass filters, leads, for the $L_1C_1L_2C_2$ type of band pass filter, to the definite integral formula

$$A_n(t)=\frac{w}{\omega_m k}\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin gx}{g}\cos 2n\mu\cdot\cos(y\sin\mu)d\mu, \qquad (3.1)$$

where $x=\omega_m t$; $y=wt/2$; $\rho=w/2\omega_m$; and

$$g=\sqrt{1+\rho^2\sin^2\mu}.$$

In solving this definite integral, use is made of the known formulas,

$$J_{2n}(y)=\frac{2}{\pi}\int_0^{\pi/2}\cos 2n\mu\cdot\cos(y\sin\mu)d\mu \qquad (3.2)$$

and

$$(-1)^s\frac{d^{2s}}{dy^{2s}}J_{2n}(y)=\frac{2}{\pi}\int_0^{\pi/2}\sin^{2s}\mu\cdot\cos 2n\mu\cdot\cos(y\sin\mu)d\mu. \qquad (3.3)$$

If in (3.1) g is replaced by unity, it follows from (3.2) that, to this approximation

$$A_n(t)=\frac{w}{\omega_m k}J_{2n}(y)\sin x \qquad (3.4)$$

which is formula (3a) of the text [28]. Clearly this becomes an increasingly good approximation as the parameter ρ becomes smaller; that is, as the ratio of the band width $w/2\pi$ to the mid-frequency $\omega_m/2\pi$ becomes smaller. The approximate formulas of the text for the other types of band pass filters were derived by precisely similar procedure and involve approximations of the same character and order of magnitude.

To investigate the approximate solution, we proceed as follows: If we write

$$\frac{\omega_m k}{w}A_n(t)=F_n(x,y)=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin gx}{g}\cos 2n\mu\cdot\cos(y\sin\mu)d\mu, \qquad (3.5)$$

and

$$G_n(x,y)=\frac{2}{\pi}\int_0^{\pi/2}\sin gx\cdot\cos 2n\mu\cdot\cos(y\sin\mu)d\mu, \qquad (3.6)$$

and if we substitute for $1/g$ in (3.5) the expansion

$$1-\frac{1}{2}\rho^2\sin^2\mu+\frac{1\cdot3}{2\cdot4}\rho^4\sin^4\mu-\ldots\ldots,$$

[28] If a series resistance R_1 and a shunt resistance $R_2=k^2/R_1$ are included in the filter sections, the formula becomes (3.4) multiplied by the factor $\exp(-R_1y/2k)$.

it follows from a formula exactly analagous to (3.3) that

$$F_n(x,y) = \left(1 + \frac{1}{2}\rho^2 \frac{\partial^2}{\partial y^2} + \frac{1\cdot 3}{2\cdot 4}\rho^4 \frac{\partial^4}{\partial y^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\rho^6 \frac{\partial^6}{\partial y^6} + \ldots\right) G_n(x,y) \quad (3.7)$$

so that the problem is reduced to the solution of the definite integral $G_n(x, y)$.

In the integral (3.6), write $g = 1 + h$, so that

$$h = \sqrt{1 + \rho^2 \sin^2\mu} - 1, \quad (3.8)$$

whence

$$G_n(x,y) = \sin x \cdot \frac{2}{\pi}\int_0^{\pi/2} \cos hx \cdot \cos 2n\mu \cdot \cos(y \sin\mu) d\mu$$
$$+ \cos x \cdot \frac{2}{\pi}\int_0^{\pi/2} \sin hx \cdot \cos 2n\mu \cdot \cos(y \sin\mu) d\mu \quad (3.9)$$

$$= P_n \sin x + Q_n \cos x, \quad (3.10)$$

where P_n and Q_n denote the definite integrals of (3.9). This effects a further reduction of the problem to the solution of the definite integrals P_n and Q_n.

In the integrands of these integrals expand $\cos hx$ and $\sin hx$ in the usual power series, and in each term thereof introduce the expansion

$$h^s = \left(\frac{\rho^2}{2}\right)^s (\sin^{2s}\mu)\,(1 + a_{s1}\rho^2 \sin^2\mu + a_{s2}\rho^4 \sin^4\mu + \ldots),$$

where the coefficients are given by

$$a_{sj} = (-1)^j s \frac{(s+2j-1)!}{(s+j)!\; j!}\left(\frac{1}{4}\right)^j.$$

By aid of this procedure it is easily shown that

$$P_n = J_{2n}(y) - \frac{(\rho^2 x/2)^2}{2!}\frac{d^4}{dy^4}\left(1 - a_{21}\rho^2 \frac{d^2}{dy^2} + a_{22}\rho^4 \frac{d^4}{dy^4} - \ldots\right) J_{2n}(y)$$
$$+ \frac{(\rho^2 x/2)^4}{4!}\frac{d^8}{dy^8}\left(1 - a_{41}\rho^2 \frac{d^2}{dy^2} + a_{42}\rho^4 \frac{d^4}{dy^4} - \ldots\right) J_{2n}(y)$$
$$- \frac{(\rho^2 x/2)^6}{6!}\frac{d^{12}}{dy^{12}}\left(1 - a_{61}\rho^2 \frac{d^2}{dy^2} + a_{62}\rho^4 \frac{d^4}{dy^4} - \ldots\right) J_{2n}(y)$$
$$+ \ldots, \quad (3.11)$$

with a corresponding expansion formula for Q_n.

It is now convenient to introduce the symbolic notation

$$P_n = \cos\left[x(\sqrt{1-\rho^2 d^2}-1)\right] J_{2n}(y) \tag{3.12}$$

and

$$Q_n = \sin\left[x(\sqrt{1-\rho^2 d^2}-1)\right] J_{2n}(y) \tag{3.13}$$

where the symbol d denotes the differential operator d/dy operating on $J_{2n}(y)$. The actual numerical significance of these formulas is gotten by expanding as in (3.11).

With the same symbolic notation we get finally,

$$A_n(t) = \frac{w}{\omega_m k} \sin\left(x\sqrt{1-\rho^2 d^2}\right) \frac{1}{\sqrt{1-\rho^2 d^2}} J_{2n}(y). \tag{3.14}$$

The exact solution (3.14) is too complicated, as it stands, to be of any practical value. Fortunately, however, it is possible to sum the expression asymptotically, and the resultant formula shows clearly the behavior of $A_n(t)$ and in particular the character and magnitude of the errors in the approximate formula of the text.

When y is large compared with $(4n)^2$,

$$\begin{aligned} J_{2n}(y) &\doteq \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{4n+1}{4}\pi\right) \\ \text{and} \\ \frac{d^{2s}}{dy^{2s}} J_{2n}(y) &\doteq (-1)^s \sqrt{\frac{2}{\pi y}} \left\{ \begin{aligned} &\left[1 - \frac{3}{2}\frac{2s(2s-1)}{4y^2}\right] \cos\left(y - \frac{4n+1}{4}\pi\right) \\ &\quad - \frac{s}{y} \sin\left(y - \frac{4n+1}{4}\pi\right) \end{aligned} \right. \end{aligned} \tag{3.15}$$

to order $1/y^2$.

If this expression is substituted in the expanded form of (3.14), some rather intricate and tedious operations finally give as the asymptotic limit of $A_n(t)$

$$A_n(t) \sim \frac{w}{\omega_m k} \sqrt{\frac{2}{\pi y}} \left\{ \begin{aligned} &\left(1 - \frac{1}{8}\rho^2 + \ldots\right) \sin\left(x\sqrt{1+\rho^2}\right) \cos\left(y - \frac{4n+1}{4}\pi\right) \\ &- \left(\frac{1}{2}\rho + \ldots\right) \cos\left(x\sqrt{1+\rho^2}\right) \sin\left(y - \frac{4n+1}{4}\pi\right). \end{aligned} \right. \tag{3.16}$$

The coefficients of the two terms of (3.16) are even and odd power series in ρ respectively, powers of ρ beyond the second being neglected.

Formula (3.16) is important, as showing the effect of the band width, that is of the parameter ρ, on the indicial admittance. It can be used for numerical computation, however, only when $y > (4n)^2$. A corresponding formula, valid over a much wider range, is obtainable from the expression derived in Appendix II for the Bessel function, namely

$$J_{2n}(y) = B_{2n}(y) \cos \Omega_{2n}(y).$$

If this expression is employed instead of (3.15), we get corresponding to (3.16),

$$A_n(t) \doteq \frac{w}{\omega_m k} B_{2n}(y) \left\{ \begin{aligned} &\left(1-\frac{1}{8}\sigma^2+\ldots\right) \sin (x\sqrt{1+\sigma^2}) \cos \Omega_{2n}(y) \\ &-\left(\frac{1}{2}\sigma+\ldots\right) \cos (x\sqrt{1+\sigma^2}) \sin \Omega_{2n}(y) \end{aligned} \right. \tag{3.17}$$

$$\doteq \frac{w}{\omega_m k} \left\{ \begin{aligned} &\left(1-\frac{1}{8}\sigma^2+\ldots\right) \sin (x\sqrt{1+\sigma^2})\, J_{2n}(y) \\ &+\left(\frac{1}{2}\rho+\ldots\right) \cos (x\sqrt{1+\sigma^2})\, J'_{2n}(y), \end{aligned} \right. \tag{3.18}$$

where $\sigma = \rho q_{2n} = \rho\sqrt{1-(2n/y)^2}$.

Formula (3.17) is valid when $y > 2n$, and ultimately approaches the limit (3.16) as y becomes indefinitely large.

We are now prepared to discuss the character of the approximations of the formula of the text, which may be written as

$$\frac{w}{2\omega_m k} B_{2n}(y) \left\{ \sin [x+\Omega_{2n}(y)] + \sin [x-\Omega_{2n}(y)] \right\}. \tag{3.19}$$

Correspondingly (3.17) may be written as

$$\frac{w}{2\omega_m k} B_{2n}(y) \left\{ \begin{aligned} &\left(1-\frac{1}{2}\sigma+\ldots\right) \sin [x\sqrt{1+\sigma^2}+\Omega_{2n}(y)] \\ +&\left(1+\frac{1}{2}\sigma+\ldots\right) \sin [x\sqrt{1+\sigma^2}-\Omega_{2n}(y)]. \end{aligned} \right. \tag{3.20}$$

Comparison of (3.19) and (3.20) shows that the approximate formula of the text ignores slowly variable correction factors in the amplitudes of the component oscillations, and a slowly variable change in their frequencies. For band pass filters employed in practice these corrections are not only slowly variable but in most cases are quite small. In any case, it is important to observe that failure to include these corrections does not appreciably affect any essential features of the building-up phenomena discussed in the text. Consequently the deductions from the formula of the text are valid not only for narrow-band pass filters, but also for filters of quite wide bands. This statement is substantiated by the fact that the steady-state characteristics, deduced from the approximate formula in accordance with the general formula V, are in excellent agreement with the exact values.

As illustrating the appropriate methods in the solution of problems in electric circuit theory, it is of interest to derive the formula for the band pass filter directly from the integral equation II. The method is not only more generally applicable, but avoids the necessity of deriving the definite integral (3.1). We therefore start with the formulas:

$$\int_0^\infty e^{-pt} A_n(t)dt = 1/p\ Z_n(p)$$

or

$$\int_0^\infty e^{-pt} A_n'(t)dt = 1/Z_n(p), \text{ where } A_n'(t) = d/dt\, A_n(t).$$

For all wave-filters of the "ladder" type it may be shown that

$$\frac{1}{Z_n(p)} = \frac{1}{z_2} \frac{(\sqrt{1+r/4} - \sqrt{r/4})^{2n}}{\sqrt{r+r^2/4}}, \tag{3.21}$$

where z_1 and z_2 are the series and shunt impedances respectively, and $r = z_1/z_2$. This expression admits of series expansion

$$\frac{1}{Z_n(p)} = \frac{2}{z_1}\left[\frac{1}{r^n} - \frac{2n+2}{1!}\frac{1}{r^{n+1}} + \frac{(2n+3)\ (2n+4)}{2!}\frac{1}{r^{n+2}} - \frac{(2n+4)\ (2n+5)\ (2n+6)}{3!}\frac{1}{r^{n+3}} + \ldots\right]. \tag{3.22}$$

For the $L_1C_1L_2C_2$ type of filter

$$1/r = \left(\frac{w}{2}\right)^2 \left(\frac{p}{p^2+\omega_m^2}\right)^2$$

and

$$1/z_1 = \frac{1}{k}\left(\frac{w}{2}\right)\left(\frac{p}{p^2+\omega_m^2}\right).$$

It follows from (3.22) and the integral identity,

$$\int_0^\infty e^{-pt} A_n'(t)dt = 1/Z_n(p)$$

that $A_n'(t)$ has an expansion solution of the form

$$A_n'(t) = \frac{w}{k}\left\{ \rho^{2n} f_{2n}(x) - \frac{2n+2}{1!}\rho^{2n+2} f_{2n+2}(x) + \frac{(2n+3)\ (2n+4)}{2!}\rho^{2n+4} f_{2n+4}(x) - \frac{(2n+4)\ (2n+5)\ (2n+6)}{3!}\rho^{2n+6} f_{2n+6}(x) \ldots\ldots \right\}, \tag{3.23}$$

where $x=\omega_m t$; $\rho=w/2\omega_m$; and the $f_s(x)$ functions are defined and determined by the integral identities,

$$\int_0^\infty f_s(x)e^{-px}dx=\left(\frac{p}{p^2+1}\right)^{s+1} \tag{3.24}$$

for all integral values of s.

For $s=0$, the solution of this equation is known; it is

$$f_0(x)=\cos x.$$

The solutions for $s>0$ are gotten from the recurrence formulas[29]

$$f_s(x)=\int_0^x \cos(x-\lambda)\, f_{s-1}(\lambda)d\lambda.$$

Repeated applications of this formula give

$$f_{2s}(x)=\frac{1}{2^{2s}}\Big(P_{2s}(x)\cos x+Q_{2s}(x)\sin x\Big)$$

where P_{2s} and Q_{2s} are polynomials in x of the $2s^{th}$ and $(2s-1)^{th}$ orders respectively. Thus:

$$P_{2s}=\alpha(s)\frac{x^{2s}}{2s!}+\beta(s)\frac{x^{2s-2}}{(2s-2)!}+\gamma(s)\frac{x^{2s-4}}{(2s-4)!}+\;.\;.$$

(terminating in term in $x^2/2!$),

and

$$Q_{2s}=a(s)\frac{x^{2s-1}}{(2s-1)!}+b(s)\frac{x^{2s-3}}{(2s-3)!}+c(s)\frac{x^{2s-5}}{(2s-5)!}+\;.\;.$$

(terminating in term in $x/1!$).

The α, β, γ . . . a, b, c, . . . coefficients are functions of the order s; the first few coefficients are:

$$\alpha(s)=1,\ s\geqq 0,$$

$$a(s)=\frac{2s+1}{2},\ s\geqq 1,$$

$$\beta(s)=-\frac{(2s-2)\ (2s+1)}{8},\ s\geqq 2,$$

$$b(s)=\frac{(2s-2)\ (2s+1)}{8}-\frac{(2s-3)\ (2s+1)\ (2s+2)}{3\cdot 16},\ s\geqq 2.$$

If the foregoing expressions for the f_s functions are substituted in the series solution (3.23) for $A_n'(t)$ and if the series are rearranged as explained below, we get writing $wt/2=\rho x=y$,

$$A_n'(t)=\frac{w}{k}\Big\{J_{2n}(y)\cos x+\rho R_1(y)\sin x+\rho^2 R_2(y)\cos x\;.\;.\Big\}.$$

[29] See equation 10, The Heaviside Operational Calculus, *B. S. T. J.*, Nov., 1922.

The first term $J_{2n}(y)$ is gotten by picking out the leading terms in the P polynomials; the second term $R_1(y)$ by picking out the leading terms in the Q polynomials; the third term $R_2(y)$, from the second terms in the P polynomials; etc.

The work of rearranging and identifying the "remainder" functions $R_1(y), R_2(y) \ldots$ is rather intricate and tedious. The first few functions can be written as

$$R_1(y)=\left(\frac{y}{2}\right)\left(\frac{d^2}{dy^2}+\frac{2}{y}\frac{d}{dy}\right)J_{2n}(y),$$

$$R_2(y)=-\frac{1}{2!}\left(\frac{y}{2}\right)^2\left(\frac{d^4}{dy^4}+\frac{4}{y}\frac{d^3}{dy^3}\right)J_{2n}(y),$$

$$R_3(y)=-\frac{1}{3!}\left(\frac{y}{2}\right)^3\left(\frac{d^6}{dy^6}+\frac{6}{y}\frac{d^5}{dy^5}-\frac{6}{y^2}\frac{d^4}{dy^4}-\frac{24}{y^3}\frac{d^3}{dy^3}\right)J_{2n}(y), \text{ etc.}$$

If we substitute these expressions, rearrange and write $\rho y/2=z$, we get finally

$$A_n'(t)=\frac{w}{k}\left\{\begin{aligned}&\cos x\left[1-\frac{z^2}{2!}\frac{d^4}{dy^4}+\frac{z^4}{4!}\frac{d^8}{dy^8}\cdots\right]J_{2n}(y)\\&+\sin x\left[\frac{z}{1!}\frac{d^2}{dy^2}-\frac{z^3}{3!}\frac{d^6}{dy^6}+\cdots\right]J_{2n}(y)\\&+\rho\sin x\left[\frac{d}{dy}-\frac{z^2}{2!}\frac{d^5}{dy^5}+\cdots\right]J_{2n}(y)\\&-\rho\cos x\left[\frac{z}{1!}\frac{d^3}{dy^3}-\frac{z^3}{3!}\frac{d^7}{dy^7}+\cdots\right]J_{2n}(y)\\&+\text{ series involving factors in } \rho^2 \text{ and higher powers.}\end{aligned}\right.$$

Neglecting factors in ρ^2, this becomes

$$A_n(t)=\frac{w}{w_m k}\left\{\begin{aligned}&\sin x\left[1-\frac{z^2}{2!}\frac{d^4}{dy^4}+\frac{z^4}{4!}\frac{d^8}{dy^8}\cdots\right]J_{2n}(y)\\&-\cos x\left[\frac{z}{1!}\frac{d^2}{dy^2}-\frac{z^3}{3!}\frac{d^6}{dy^6}+\frac{z^5}{5!}\frac{d^{10}}{dy^{10}}\cdots\right]J_{2n}(y).\end{aligned}\right.$$

The character of this solution in the region $y>2n$, is shown by the asymptotic approximation

$$A_n(t)=\frac{w}{\omega_m k}J_{2n}(y)\sin\left(1+\tfrac{1}{2}\rho^2 q_{2n}^2\right)x \tag{3.25}$$

where

$$q_{2n}=\sqrt{1-\frac{(2n)^2}{y^2}}.$$

To the same order of approximation in $\rho = w/2\omega_m$, this agrees with the solution (3.18) given above.

Appendix II

Properties of the Bessel Function $J_n(x)$

The Bessel functions have been studied and tabulated more exhaustively than any other functions largely owing to their great importance and frequent occurrence in mathematical physics. Qualitatively their behavior for integral orders n and real arguments x may be described as follows.

When the argument is less than the order ($0 \leqq x < n$) the function is very small and positive, and is initially zero (except when $n=0$). In the neighborhood of $x=n$, the function begins to build up and reaches a maximum a little beyond the point $x=n$. Thereafter the function oscillates with increasing frequency and diminishing amplitude, and ultimately behaves as

$$\sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{2n+1}{4}\pi\right).$$

When $n=0$, the initial value is unity, but the subsequent behavior of the function is as described above.

In order to get a more accurate picture of this function the following approximate formula was developed in the course of the present investigation.[30]

$$J_n(x) \doteq B_n(x) \cos \Omega_n(x), \qquad \text{for } x > n$$

where

$$B_n(x) = \sqrt{\frac{2}{\pi x}} \frac{1}{\left(1 - \frac{m^2}{x^2} + \frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}\right)^{1/4}},$$

$$\Omega_n(x) = x\left[\sqrt{1 - \frac{m^2}{x^2}} + \frac{m}{x}\sin^{-1}\left(\frac{m}{x}\right) - \frac{m^2}{4x^4}\frac{1}{(1-m^2/x^2)^{3/2}}\right] - \frac{2n+1}{4}\pi,$$

$$\Omega_n'(x) = \frac{d}{dx}\Omega_n(x),$$

$$= \sqrt{1 - \frac{m^2}{x^2} + \frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}},$$

and

$$m^2 = n^2 - 1/4.$$

[30] It was subsequently discovered that somewhat similar formulas had previously been developed by Graf and Gubler (Einleitung in die Theorie der Besselschen Funktionen), and by Nicholson (*Phil. Mag.*, 1910, p. 249).

This approximate formula is valid only where $x>n$, its accuracy increasing with x and with n. For all orders of n it is quite accurate beyond the first zero of the function.

The "instantaneous frequency" of oscillation is approximately

$$\frac{1}{2\pi}\Omega_n'(x)=\frac{1}{2\pi}\sqrt{1-\frac{m^2}{x^2}+\frac{3}{2}\frac{m^2}{x^4}\frac{1}{(1-m^2/x^2)^2}}.$$

By this it is meant that at any point x ($>n$) the interval between successive zeros is approximately $\pi/\Omega'(x)$. Otherwise stated, in the neighborhood of any point x, the function behaves like a sinusoid of amplitude $B_n(x)$ and frequency $\omega/2\pi$ where $\omega=\Omega_n'(x)$.

The following approximate formulas, while not sufficiently precise for the purposes of accurate computation except for quite large values of x, clearly exhibit the character of the functions for values of the argument $x>n$, and of the order $n>2$.

$$J_n(x)\doteq h_n\sqrt{\frac{2}{\pi x}}\cos\,(q_nx-\Theta_n),$$

$$J_n'(x)=-q_nh_n\sqrt{\frac{2}{\pi x}}\sin\,(q_nx-\Theta_n),$$

$$\int_0^x J_n\,(x)dx=1+\frac{h_n}{q}\sqrt{\frac{2}{\pi x}}\sin\,(q_nx-\Theta_n),$$

where

$$h_n=\left(\frac{1}{1-n^2/x^2}\right)^{1/4}\doteq 1+\frac{n^2}{4x^2},$$

$$q_n=\sqrt{1-n^2/x^2},$$

and

$$\Theta_n=\frac{2n+1}{4}\pi-n\,\sin^{-1}(n/x).$$

Appendix III

Building-Up of Alternating Currents in Wave-Filters

The integrals

$$S_n(z;\nu)=\int_0^z J_n(z_1)\,\sin\,\nu(z-z_1)dz_1$$

and

$$C_n(z;\nu)=\int_0^z J_n(z_1)\,\cos\,\nu(z-z_1)dz_1,$$

on which the genesis and growth of alternating currents in the low pass and band pass filters depends, have been computed as follows.

For values of $z<24$, $n\leqq 10$ and $\nu\leqq 1$, they are accurately calculable from the following series expansions

$$C_n(z;\nu)=2(c_1J_{n+1}(z)+c_3J_{n+3}(z)+c_5J_{n+5}(z)+\ .\ .),$$

and

$$S_n(z;\nu)=4\nu(c_2J_{n+2}(z)+c_4J_{n+4}(z)+c_6J_{n+6}(z)+\ .\ .),$$

where the coefficients c_1, c_2 . . . are polynomials in 2ν, and are independent of the index n. They are

$$c_1=1,$$

$$c_3=1-(2\nu)^2,$$

$$c_5=1-\frac{3}{1!}(2\nu)^2+(2\nu)^4,$$

$$c_7=1-\frac{3\cdot 4}{2!}(2\nu)^2+\frac{5}{1!}(2\nu)^4-(2\nu)^6,$$

$$c_9=1-\frac{3\cdot 4\cdot 5}{3!}(2\nu)^2+\frac{5\cdot 6}{2!}(2\nu)^4-\frac{7}{2!}(2\nu)^6+(2\nu)^8,$$

. . . .

$$c_2=1,$$

$$c_4=\frac{2}{1!}-(2\nu)^2,$$

$$c_6=\frac{2\cdot 3}{2!}-\frac{4}{1!}(2\nu)^2+(2\nu)^4,$$

$$c_8=\frac{2\cdot 3\cdot 4}{3!}-\frac{4\cdot 5}{2!}(2\nu)^2+\frac{6}{1!}(2\nu)^4-(2\nu)^6,$$

. . . .

The tabulation of $J_n(z)$ for values of z up to 24 and of n up to 60 given by Gray and Mathews and by Jahnke und Emde make the computation for integral values of z rapid and precise.

For large values of n the integrals can be accurately computed, except in the neighborhood of the critical point $z=n/\sqrt{1-\nu^2}$, $(\nu<1)$, from the asymptotic formulas furnished by Gronwall.

Without detailed computation, however, the general character of the integrals can be shown as follows with an accuracy usually sufficient for engineering purposes. By differentiation S_n and C_n satisfy the differential equations

$$S_n'=\nu C_n,$$

and

$$C_n'=J_n(z)-\nu S_n,$$

where the primes denote differentiation with respect to the argument z. The solution of these differential equations is based on the approximation, valid only when $z>n$,

$$\frac{d^2}{dz^2}J_n(z) \doteq -q_n^2 J_n(z), \qquad q_n=\sqrt{1-n^2/z^2}.$$

To this approximation, which becomes more and more accurate as z and n increase, the differential equations are satisfied by solutions of the form

$$S_n=\frac{\nu}{\nu^2-q_n^2}J_n(z)+A\sin(\nu z-\alpha),$$

and

$$C_n=\frac{1}{\nu^2-q_n^2}J_n'(z)+A\cos(\nu z-\alpha).$$

A and α in the complementary terms are arbitrary constants, which must be determined. These complementary terms, periodic in νz, are evidently the ultimate values of the integrals when z approaches infinity, which are known. Other considerations, however, show that these terms should be omitted when $\nu<1$ and $z<n/\sqrt{1-\nu^2}$. Consequently we arrive at the following approximations.[31]

For $\nu<1$ and $n<z<n/\sqrt{1-\nu^2}$,

$$S_n(z;\nu)=\frac{\nu}{\nu^2-q_n^2}J_n(z),$$

$$C_n(z;\nu)=\frac{1}{\nu^2-q_n^2}J_n'(z),$$

and

$$q_n=\sqrt{1-n^2/z^2}.$$

This approximation is not accurate at $z=n$, and breaks down at the critical point $z=n/\sqrt{1-\nu^2}$. In the interval between, however, it is a fair approximation, particularly when ν is nearly equal to unity and n is not too small.

For $\nu<1$ and $z>n/\sqrt{1-\nu^2}$,

$$S_n(z;\nu)=\frac{\nu}{\nu^2-q_n^2}J_n(z)+\frac{1}{\sqrt{1-\nu^2}}\sin(\nu z-n\sin^{-1}\nu),$$

and

$$C_n(z;\nu)=\frac{1}{\nu^2-q_n^2}J_n'(z)+\frac{1}{\sqrt{1-\nu^2}}\cos(\nu z-n\sin^{-1}\nu).$$

[31] The qualitative properties of these definite integrals can be deduced from the principle of stationary phase (See Theory of Bessel Functions, G. N. Watson, p. 229).

This formula can be safely employed only when z considerably exceeds the critical value $n/\sqrt{1-\nu^2}$.

For $\nu>1$ *and* $z>n$, the ultimate periodic terms are very small, and may be omitted unless n is too small. Consequently in this region,

$$S_n(z;\nu) \doteq \frac{\nu}{\nu^2-q_n^2} J_n(z),$$

and

$$C_n(z;\nu) \doteq \frac{1}{\nu^2-q_n^2} J'_n(z).$$

In the range of values for which the foregoing approximations are valid we have also to the same approximation (see Appendix II)

$$J_n(z) \doteq \sqrt{\frac{2}{\pi z}} \cos\,(q_n z - \Theta_n),$$

and

$$J'_n(z) \doteq -q_n \sqrt{\frac{2}{\pi z}} \sin\,(q_n z - \Theta_n).$$

Appendix IV

The Effects of Terminal Impedances

In the text of this paper, the calculation of the wave-filter indicial admittances is based on the assumption that the voltage is applied directly to the filter at "mid-series" position and that the filter is either infinitely long or else, what amounts to the same thing, is terminated in its characteristic impedance. By virtue of these assumptions, the disturbing effects of terminal reflections are eliminated, and, as shown in the text, the solution is reducible to a relatively simple form, which admits of considerable instructive interpretation by inspection, and is rather easily computed.

In the following the general solution will be given for the indicial admittance $A_n(t)$ in the nth section of a wave-filter of s sections or length, with the e.m.f. applied to the initial or zero-th section through an impedance $Z_1(p)=Z_1$ and the last or sth section closed by an impedance $Z_2(p)=Z_2$.

For any type of periodic structure, including as a limiting case, the smooth line, it can readily be shown that

$$\frac{1}{Z_n(p)} = \sigma \frac{1}{K_1} \frac{e^{-n\Gamma}+\rho_2 e^{-(2s-n)\Gamma}}{1-\rho_1\rho_2 e^{-2s\Gamma}} \tag{1}$$

where

K_1 = characteristic impedance, as seen from terminals of initial or zero-th section,

K_2 = characteristic impedance, as seen from terminals of last or sth section,

Γ = propagation constant per section,

Z_1, Z_2 = terminal impedances,

$$\sigma = \frac{K_1}{K_1+Z_1},$$

$$\rho_1 = \frac{K_1-Z_1}{K_1+Z_1},$$

and

$$\rho_2 = \frac{K_2-Z_2}{K_2+Z_2}.$$

K_1, K_2, Z_1, Z_2, and consequently σ, ρ_1, ρ_2 are, of course, functions of the operator p.

The corresponding indicial admittance $A_n(t)$ is given by the integral equation

$$\int_0^\infty e^{-pt} A_n(t) dt = \frac{1}{pZ_n(p)}. \tag{2}$$

By aid of (1) the right hand side of (2) can be expanded as

$$\sigma \frac{e^{-n\Gamma}}{pK_1} + \sigma\rho_2 \frac{e^{-(2s-n)\Gamma}}{pK_1} + \sigma\rho_1\rho_2 \frac{e^{-(2s+n)\Gamma}}{pK_1} + \sigma\rho_1\rho_2^2 \frac{e^{-(4s-n)\Gamma}}{pK_1} + \sigma\rho_1^2\rho_2^2 \frac{e^{-(4s+n)\Gamma}}{pK_1} + \ldots\ldots. \tag{3}$$

Now if $a_m(t)$ denotes the indicial admittance in the mth section of an infinitely long periodic structure, when the e.m.f. is applied directly to the sending end terminals, it follows from (2) and (3) that

$$\int_0^\infty e^{-pt} a_m(t) dt = \frac{e^{-m\Gamma}}{pK_1}. \tag{4}$$

From (2), (3) and (4) it follows at once that

$$A_n(t) = \frac{d}{dt}\int_0^t dy \left\{ \begin{array}{l} r_0(t-y)a_n(y) + r_1(t-y)a_{2s-n}(y) + r_2(t-y)a_{2s+n}(y) \\ + r_3(t-y)a_{4s-n}(y) + r_4(t-y)a_{4s+n}(y) + \dots \end{array} \right. \qquad (5)$$

provided the functions $r_0(t)$, $r_1(t)$, $r_2(t)$ . . . satisfy, and are defined by, the equations

$$\int_0^\infty e^{-pt} r_0(t)dt = \frac{\sigma}{p} = \frac{1}{p}\frac{K_1}{K_1+Z_1},$$

$$\int_0^\infty e^{-pt} r_1(t)dt = \frac{\sigma\rho_2}{p} = \frac{1}{p}\frac{K_1}{K_1+Z_1}\cdot\frac{K_2-Z_2}{K_2+Z_2}, \qquad (6)$$

$$\int_0^\infty e^{-pt} r_2(t)dt = \frac{\sigma\rho_1\rho_2}{p} = \frac{1}{p}\frac{K_1}{K_1+Z_1}\cdot\frac{K_1-Z_1}{K_1+Z_1}\cdot\frac{K_2-Z_2}{K_2+Z_2}, \text{ etc.}$$

If the indicial admittance in any section of an infinitely long periodic structure is determined, and equations (6) solved for $r_0(t)$,$r_1(t)$, $r_2(t)$ . . . (by aid of any of the methods discussed in the present paper), then $A_n(t)$ is given by (5) by a single quadrature. The solution may appear quite involved; as a matter of fact it is the simplest and most easily interpreted and computed form of solution possible and its complexity merely reflects the complicated character of reflection effects due to terminal impedances. This considered statement is made in the light of an extensive study of the whole problem and the literature bearing on it and has been tested in many specific cases.

When the terminal impedances Z_1 and Z_2 are complicated and entirely unrelated to the corresponding characteristic impedances K_1 and K_2, the solution of equations (6) and the numerical computations of (5) are laborious but entirely possible, the only questions being as to whether the importance of the problem justifies the necessary expenditure of time and effort. In many cases, also, approximate solutions are obtainable. Without any computations, however, the solution (5) admits of considerable instructive interpretation by inspection. The first term represents the current in the nth section of an infinitely long structure when a unit e.m.f. is impressed through a terminal impedance Z_1. $r_0(t)$ is the corresponding voltage which exists across the terminals proper. The second term is a reflected wave from the other terminals due to the terminal impedance irregularity which exists there. The third term is a reflected wave from the sending end terminals due to the corresponding terminal impedance irregularity, etc. The solution, consequently, is expanded in a form which corresponds exactly with the actual sequence of phenomena which occur.

The solution takes a particularly simple and instructive form when $Z_1 = k_1 K_1$ and $Z_2 = k_2 K_2$ where k_1 and k_2 are numerics. In this case the solutions of (6) give

$$r_0(t) = r_0 = \frac{1}{1+k_1},$$

$$r_1 = \frac{1}{1+k_1} \cdot \frac{1-k_2}{1+k_2},$$

$$r_2 = \frac{1}{1+k_1} \cdot \frac{1-k_1}{1+k_1} \cdot \frac{1-k_2}{1+k_2}, \quad \text{etc. and}$$

$$A_n(t) = \frac{1}{1+k_1} \left\{ a_n(t) + \frac{1-k_2}{1+k_2} a_{2s-n}(t) + \frac{1-k_1}{1+k_1} \cdot \frac{1-k_2}{1+k_2} a_{2s+n}(t) + \dots \right\}.$$

The solution for the special cases of open and short circuit terminations follow at once by assigning the values of zero or infinity, as the case may be, to k_1 and k_2. If $k_1 = o$; $k_2 = 1$, $A_n(t)$ reduces to $a_n(t)$ as, of course, it should.

Low Pass Wave-Filter, Type L_1C_2

Divide ordinates by k and abscissae by ω_c to read current in amperes and time in seconds.

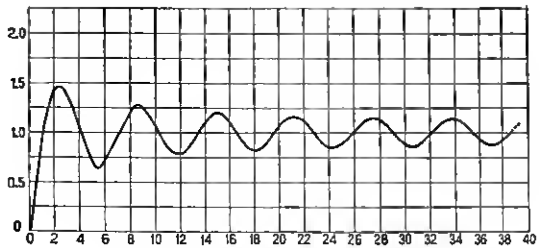

Fig. 8

Indicial Admittance of Initial Section.

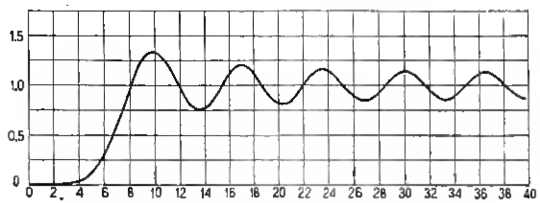

Fig. 9

Indicial Admittance of Third Section.

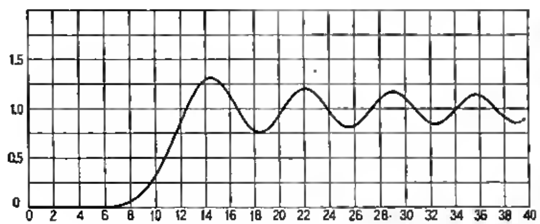

Fig. 10

Indicial Admittance of Fifth Section.

Band Pass Wave-Filter, Type $L_1C_1L_2C_2$

Divide ordinates by $\omega_m k/w$ and abscissae by $w/2$ to read current in amperes and time in seconds.

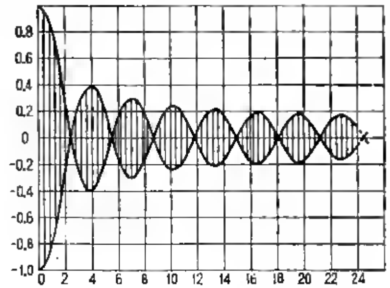

Fig. 11

Indicial Admittance of Initial Section.

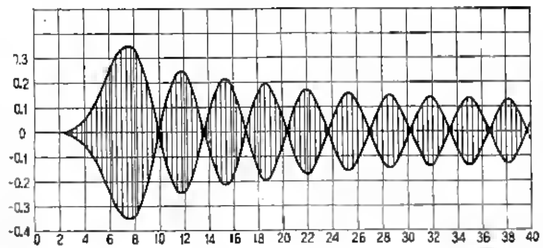

Fig. 12

Indicial Admittance of Third Section.

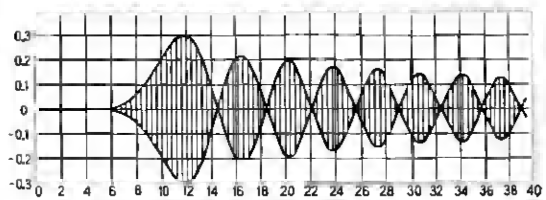

Fig. 13

Indicial Admittance of Fifth Section.

Band Pass Wave-Filter, Type $L_1L_2C_2$

Divide ordinates by $\omega_m k/2w$ and abscissae by $w/2$ to read current in amperes and time in seconds.

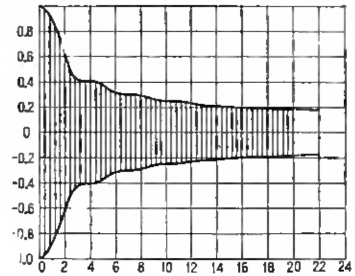

Fig. 14

Indicial Admittance of Initial Section.

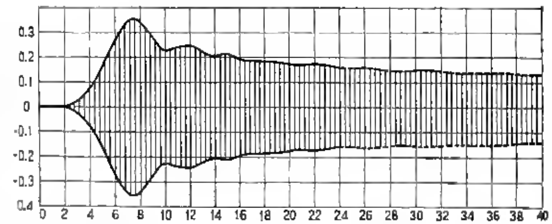

Fig. 15

Indicial Admittance of Sixth Section.

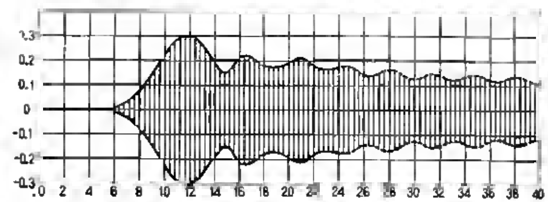

Fig. 16

Indicial Admittance of Tenth Section.

High Pass Wave-Filter, Type C_1L_2

Divide ordinates by k and abscissae by ω_c to read current in amperes and time in seconds.

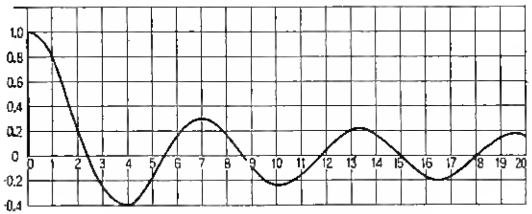

Fig. 17

Indicial Admittance of Initial Section.

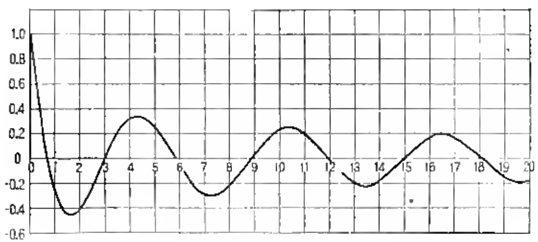

Fig. 18

Indicial Admittance of First Section.

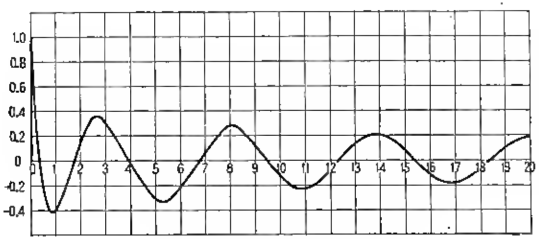

Fig. 19

Indicial Admittance of Second Section.

High Pass Wave-Filter, Type C_1L_2—*Contd.*

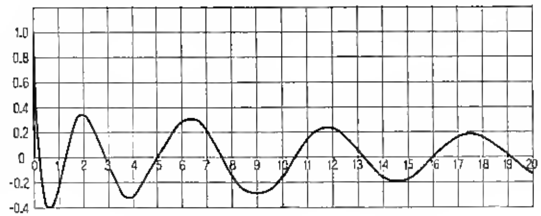

Fig. 20

Indicial Admittance of Third Section.

Low Pass Wave-Filter, Type L_1C_2

Divide ordinates by k and abscissae by ω_c to read current in amperes and time in seconds.

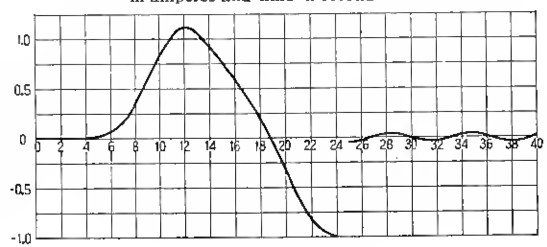

Fig. 21

Current in Third Section in Response to E.M.F. sin $\frac{1}{4}\omega_c t$.

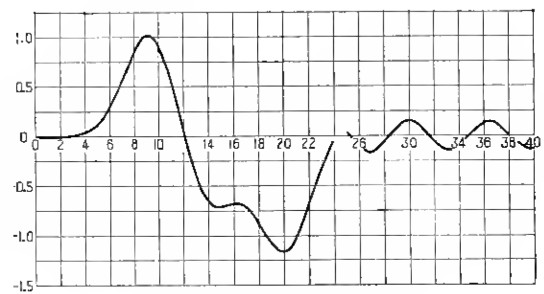

Fig. 22

Current in Third Section in Response to E.M.F. cos $\frac{1}{4}\omega_c t$.

Low Pass Wave-Filter, Type L_1C_2—*Contd.*

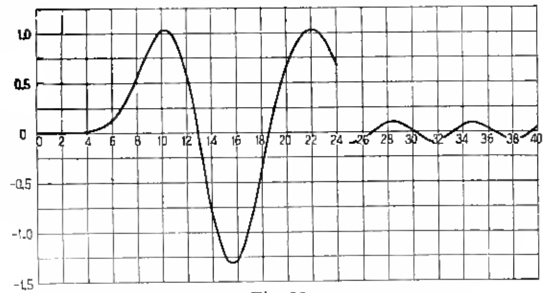

Fig. 23

Current in Third Section in Response to E.M.F. sin $\frac{1}{2}\omega_c t$.

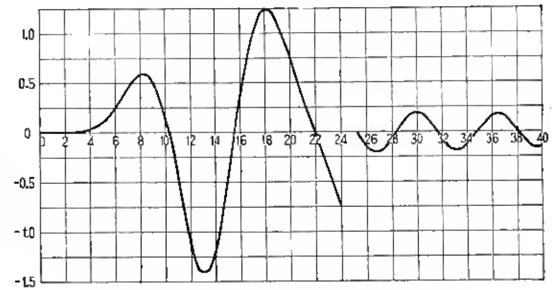

Fig. 24

Current in Third Section in Response to E.M.F. cos $\frac{1}{2}\omega_c t$.

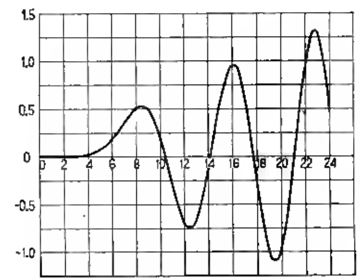

Fig. 25

Current in Third Section in Response to E.M.F. sin $\omega_c t$.

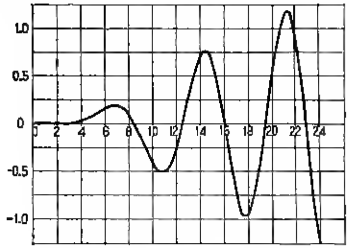

Fig. 26
Current in Third Section in Response to E.M.F. cos $\omega_c t$.

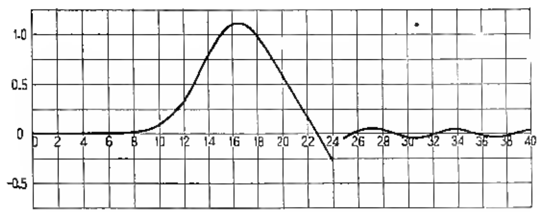

Fig. 27
Current in Fifth Section in Response to E.M.F. sin $\frac{1}{4}\omega_c t$.

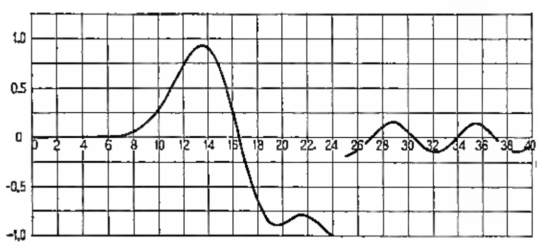

Fig. 28
Current in Fifth Section in Response to E.M.F. cos $\frac{1}{4}\omega_c t$.

Low Pass Wave-Filter, Type L_1C_2—*Contd.*

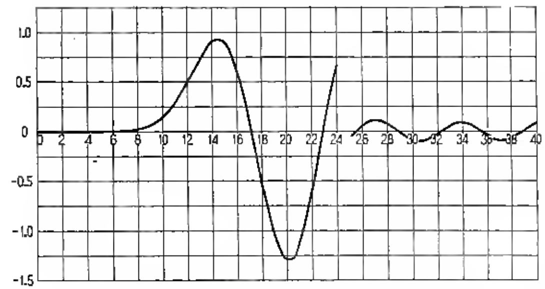

Fig. 29
Current in Fifth Section in Response to E.M.F. sin $\frac{1}{2}\omega_c t$.

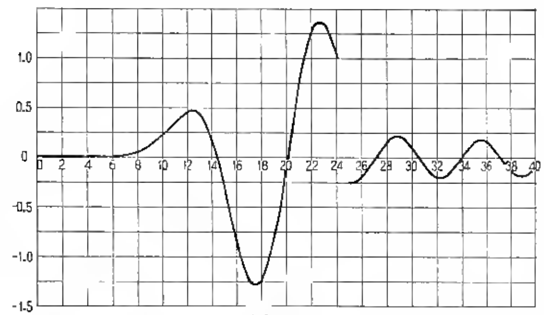

Fig. 30
Current in Fifth Section in Response to E.M.F. cos $\frac{1}{2}\omega_c t$.

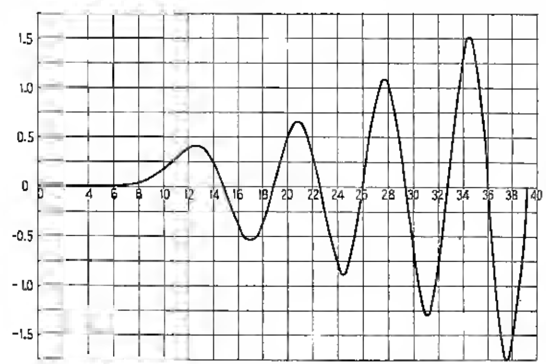

Fig. 31
Current in Fifth Section in Response to E.M.F. sin $\omega_c t$.

Low Pass Wave-Filter, Type L_1C_2—*Contd.*

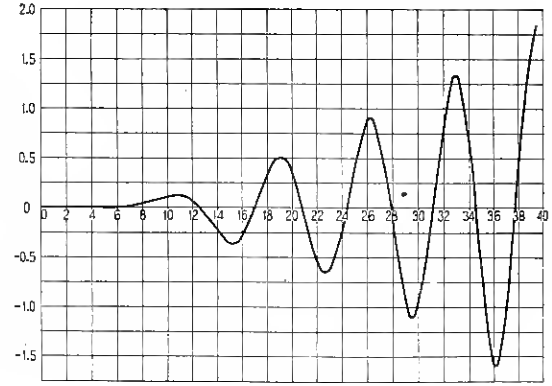

Fig. 32
Current in Fifth Section in Response to E.M.F. cos $\omega_c t$.

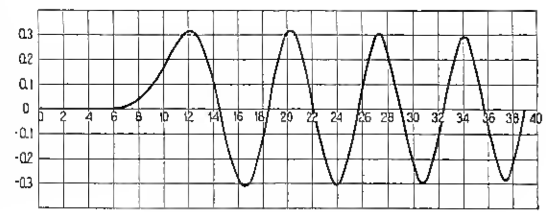

Fig. 33
Current in Fifth Section in Response to E.M.F. sin 1.25 $\omega_c t$.
Steady-state Amplitude=0.0013.

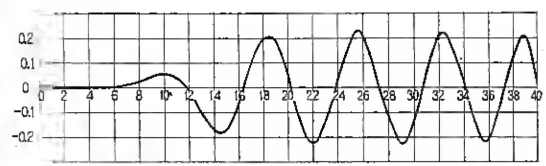

Fig. 34
Current in Fifth Section in Response to E.M.F. cos 1.25 $\omega_c t$.
Steady-state Amplitude=0.0013.

Band Pass Wave-Filter, Type $L_1C_1L_2C_2$

Divide ordinates by $\omega_m k/w$ and abscissae by $w/2$ to read current in amperes and time in seconds.

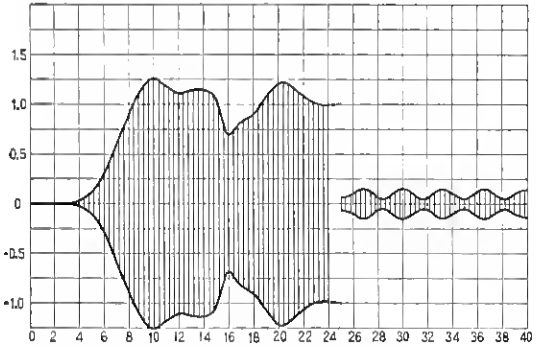

Fig. 35

Envelope of Current in Third Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/8)$.

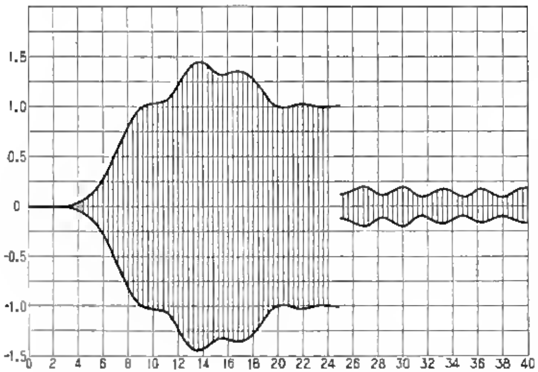

Fig. 36

Envelope of Current in Third Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/4)$.

Band Pass Wave-Filter, Type $L_1C_1L_2C_2$, *Contd.*

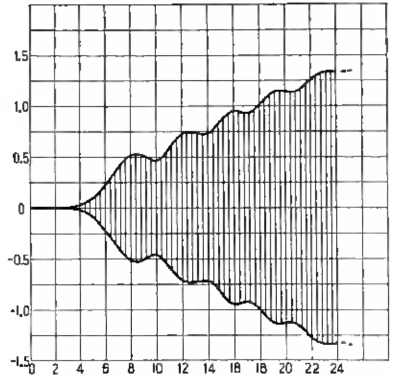

Fig. 37

Envelope of Current in Third Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/2)$.

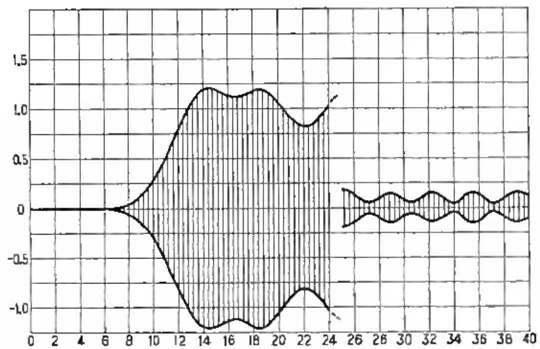

Fig. 38

Envelope of Current in Fifth Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/8)$.

Band Pass Wave-Filter, Type $L_1C_1L_2C_2$, *Contd.*

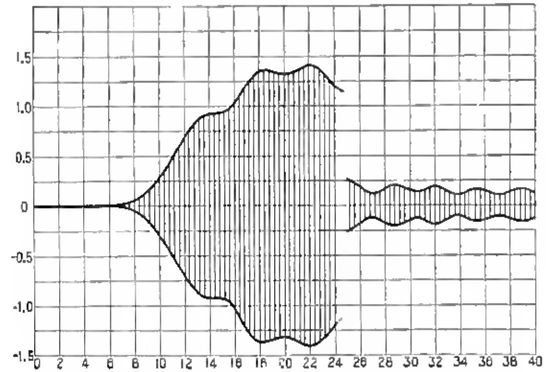

Fig. 39

Envelope of Current in Fifth Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/4)$.

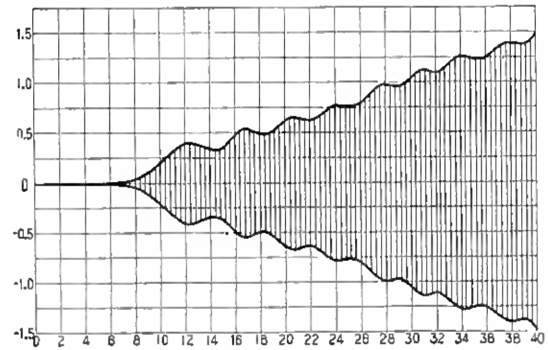

Fig. 40

Envelope of Current in Fifth Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/2)$.

Band Pass Wave-Filter, Type $L_1C_1L_2C_2$, *Contd.*

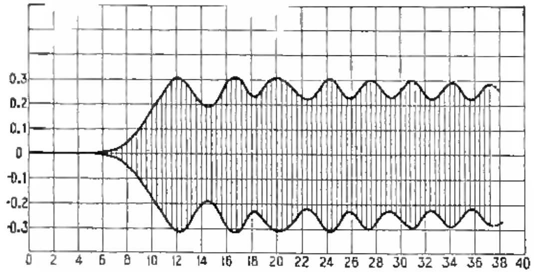

Fig. 41

Envelope of Current in Fifth Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}\left(\omega_m \pm \frac{5}{4}w/2\right)$.

Band Pass Wave-Filter, Type $L_1L_2C_2$

Divide ordinates by $\omega_m k/^{-}w$ and abscissae by $w/2$ to read current in amperes and time in seconds.

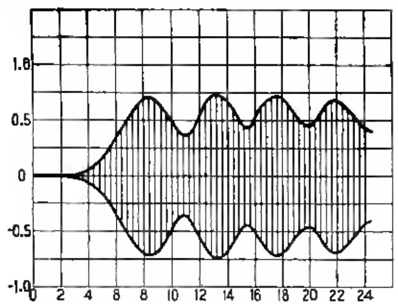

Fig. 42

Envelope of Current in Sixth Section in Response to E.M.F. of Frequency $\frac{1}{2\pi}(\omega_m \pm w/4)$.

High Pass Wave-Filter, Type C_1L_2

Divide ordinates by k and abscissae by ω_c to read current in amperes and time in seconds.

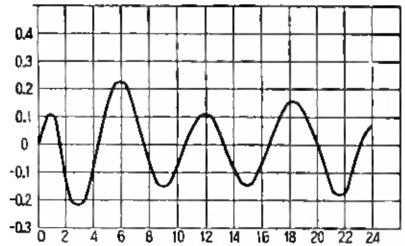

Fig. 43

[Current in First Section in Response to E.M.F. sin $\frac{1}{2}\omega_c t$.
Steady-state Amplitude=0.0415.

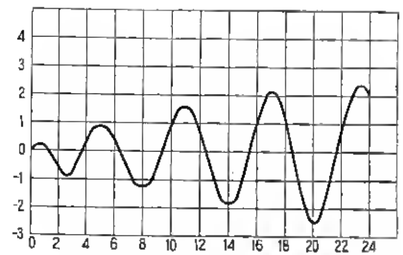

Fig. 44

Current in First Section in Response to E.M.F. sin $\omega_c t$.