Mathematical Analysis of Random Noise

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Introduction

THIS paper deals with the mathematical analysis of noise obtained by passing random noise through physical devices. The random noise considered is that which arises from shot effect in vacuum tubes or from thermal agitation of electrons in resistors. Our main interest is in the statistical properties of such noise and we leave to one side many physical results of which Nyquist's law may be given as an example.[1]

About half of the work given here is believed to be new, the bulk of the new results appearing in Parts III and IV. In order to provide a suitable introduction to these results and also to bring out their relation to the work of others, this paper is written as an exposition of the subject indicated in the title.

When a broad band of random noise is applied to some physical device, such as an electrical network, the statistical properties of the output are often of interest. For example, when the noise is due to shot effect, its mean and standard deviations are given by Campbell's theorem (Part I) when the physical device is linear. Additional information of this sort is given by the (auto) correlation function which is a rough measure of the dependence of values of the output separated by a fixed time interval.

The paper consists of four main parts. The first part is concerned with shot effect. The shot effect is important not only in its own right but also because it is a typical source of noise. The Fourier series representation of a noise current, which is used extensively in the following parts, may be obtained from the relatively simple concepts inherent in the shot effect.

The second part is devoted principally to the fundamental result that the power spectrum of a noise current is the Fourier transform of its correlation function. This result is used again and again in Parts III and IV.

A rather thorough discussion of the statistics of random noise currents is given in Part III. Probability distributions associated with the maxima of the current and the maxima of its envelope are developed. Formulas for the expected number of zeros and maxima per second are given, and a start is made towards obtaining the probability distribution of the zeros.

When a noise voltage or a noise voltage plus a signal is applied to a non-

[1] An account of this field is given by E. B. Moullin, "Spontaneous Fluctuations of Voltage," Oxford (1938).

linear device, such as a square-law or linear rectifier, the output will also contain noise. The methods which are available for computing the amount of noise and its spectral distribution are discussed in Part IV.

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SUMMARY OF RESULTS

Before proceeding to the main body of the paper, we shall state some of the principal results. It is hoped that this summary will give the casual reader an over-all view of the material covered and at the same time guide the reader who is interested in obtaining some particular item of information to those portions of the paper whicb may possibly contain it.

Part I—Shot Effect

Shot effect noise results from the superposition of a great number of disturbances which occur at random. A large class of noise generators produce noise in this way.

Suppose that the arrival of an electron at the anode of the vacuum tube at time $t = 0$ produces an effect $F(t)$ at some point in the output circuit. If the output circuit is such that the effects of the various electrons add linearly, the total effect at time t due to all the electrons is

$$I(t) = \sum_{k=-\infty}^{+\infty} F(t - t_k) \tag{1.2–1}$$

where the k^{th} electron arrives at t_k and the series is assumed to converge. Although the terminology suggests that $I(t)$ is a current, and it will be spoken of as a noise current, it may be any quantity expressible in the form (1.2–1).

1. Campbell's theorem: The average value of $I(t)$ is

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.2–2}$$

and the mean square value of the fluctuation about this average is

$$\text{ave.}\, [I(t) - \overline{I(t)}]^2 = \nu \int_{-\infty}^{+\infty} F^2(t)\, dt \tag{1.2–3}$$

where ν is the average number of electrons arriving per second at the anode. In this expression the electrons are supposed to arrive independently and at random. $\nu e^{-\nu t}\, dt$ is the probability that the length of the interval between two successive arrivals lies between t and $t + dt$.

2. Generalization of Campbell's theorem. Campbell's theorem gives information about the average value and the standard deviation of the probability distribution of $I(t)$. A generalization of the theorem gives information about the third and higher order moments. Let

$$I(t) = \sum_{-\infty}^{+\infty} a_k F(t - t_k) \tag{1.5-1}$$

where $F(t)$ and t_k are of the same nature as those in (1.2–1) and $\cdots a_1$, $a_2, \cdots a_k, \cdots$ are independent random variables all having the same distribution. Then the n^{th} semi-invariant of the probability density $P(I)$ of $I = I(t)$ is

$$\lambda_n = \nu \overline{a^n} \int_{-\infty}^{+\infty} [F(t)]^n \, dt \tag{1.5-2}$$

The semi-invariants are defined as the coefficients in the expansion of the characteristic function $f(u)$:

$$\log_e f(u) = \sum_{n=1} \frac{\lambda_n}{n!} (iu)^n \tag{1.5-3}$$

where

$$f(u) = \text{ave. } e^{iIu} = \int_{-\infty}^{+\infty} P(I) e^{iIu} \, dI$$

The moments may be computed from the λ's.

3. As $\nu \to \infty$ the probability density $P(I)$ of the shot effect current approaches a normal law. The way it is approached is given by

$$P(I) \sim \sigma^{-1} \varphi^{(0)}(x) - \frac{\lambda_3 \sigma^{-4}}{3!} \varphi^{(3)}(x) + \left[\frac{\lambda_4 \sigma^{-5}}{4!} \varphi^{(4)}(x) + \frac{\lambda_3^2 \sigma^{-7}}{72} \varphi^{(6)}(x) \right] + \cdots \tag{1.6-3}$$

where the λ's are given by (1.5–2) and

$$\sigma^2 = \lambda_2 \qquad x = \frac{I - \bar{I}}{\sigma} \qquad \varphi^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dx^n} e^{-x^2/2}$$

Since the λ's are of the order of ν, σ is of the order of $\nu^{1/2}$ and the orders of σ^{-1}, $\lambda_3\sigma^{-4}$, $\lambda_4\sigma^{-5}$ and $\lambda_3^2\sigma^{-7}$ are $\nu^{-1/2}$, ν^{-1}, $\nu^{-3/2}$ and $\nu^{-3/2}$ respectively. A possible use of this result is to determine whether a noise due to random independent events occuring at the rate of ν per second may be regarded as "random noise" in the sense of this work.

4. When $I(t)$, as given by (1.5–1), is analyzed as a Fourier series over an interval of length T a set of Fourier coefficients is obtained. By taking many different intervals, all of length T, many sets of coefficients are obtained. If ν is sufficiently large these coefficients tend to be distributed normally and independently. A discussion of this is given in section 1.7.

Part II—Power Spectra and Correlation Functions

1. Suppose we have a curve, such as an oscillogram of a noise current, which extends from $t = 0$ to $t = \infty$. Let this curve be denoted by $I(t)$. The correlation function of $I(t)$ is $\psi(\tau)$ which is defined as

$$\psi(\tau) = \operatorname*{Limit}_{T\to\infty} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \tag{2.1–4}$$

where the limit is assumed to exist. This function is closely connected with another function, the power spectrum, $w(f)$, of $I(t)$. $I(t)$ may be regarded as composed of many sinusoidal components. If $I(t)$ were a noise current and if it were to flow through a resistance of one ohm the average power dissipated by those components whose frequencies lie between f and $f + df$ would be $w(f)\, df$.

The relation between $w(f)$ and $\psi(\tau)$ is

$$w(f) = 4 \int_0^\infty \psi(\tau) \cos 2\pi f\tau\, d\tau \tag{2.1–5}$$

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1–6}$$

When $I(t)$ has no d.c. or periodic components,

$$w(f) = \operatorname*{Limit}_{T\to\infty} \frac{2\,|\,S(f)\,|^2}{T} \tag{2.1–3}$$

where

$$S(f) = \int_0^T I(t)e^{-2\pi i f t}\, dt.$$

The correlation function for

$$I(t) = A + C \cos (2\pi f_0 t - \varphi)$$

is

$$\psi(\tau) = A^2 + \frac{C^2}{2} \cos 2\pi f_0 \tau \tag{2.2–3}$$

These results are discussed in sections 2.1 to 2.4 inclusive.

2. So far we have supposed $I(t)$ to be some definite function for which a curve may be drawn. Now consider $I(t)$ to be given by a mathematical expression into which, besides t, a number of parameters enter. $w(f)$ and $\psi(\tau)$ are now obtained by averaging the integrals over the possible values of the parameters. This is discussed in section 2.5.

3. The correlation function for the shot effect current of (1.2–1) is

$$\psi(\tau) = \nu \int_{-\infty}^{+\infty} F(t)F(t+\tau)\,dt + \left[\nu \int_{-\infty}^{+\infty} F(t)\,dt\right]^2 \qquad (2.6\text{–}2)$$

The distributed portion of the power spectrum is

$$w_1(f) = 2\nu\,|\,s(f)\,|^2$$

where

$$s(f) = \int_{-\infty}^{+\infty} F(t)e^{-2\pi ift}\,dt \qquad (2.6\text{–}5)$$

The complete power spectrum has in addition to $w_1(f)$ an impulse function representing the d.c. component $\bar{I}(t)$.

In the formulas above for the shot effect it was assumed that the expected number, ν, of electrons per second did not vary with time. A case in which ν does vary with time is briefly discussed near the end of Section 2.6.

4. Random telegraph signal. Let $I(t)$ be equal to either a or $-a$ so that it is of the form of a flat top wave, and let the lengths of the tops and bottoms be distributed independently and exponentially. The correlation function and power spectrum of I are

$$\psi(\tau) = a^2 e^{-2\mu|\tau|} \qquad (2.7\text{–}4)$$

$$w(f) = \frac{2a^2\mu}{\pi^2 f^2 + \mu^2} \qquad (2.7\text{–}5)$$

where μ is the expected number of changes of sign per second.

Another type of random telegrapb signal may be formed as follows: Divide the time scale into intervals of equal length h. In an interval selected at random the value of $I(t)$ is independent of the value in the other intervals and is equally likely to be $+a$ or $-a$. The correlation function of $I(t)$ is zero for $|\,\tau\,| > h$ and is

$$a^2\left(1 - \frac{|\,\tau\,|}{h}\right)$$

for $0 \leq |\,\tau\,| < h$ and the power spectrum is

$$w(f) = 2h\left(\frac{a \sin \pi fh}{\pi fh}\right)^2 \qquad (2.7\text{–}9)$$

5. There are two representations of a random noise current which are especially useful. The first one is

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \qquad (2.8\text{–}1)$$

where a_n and b_n are independent random variables whicb are distributed normally about zero with the standard deviation $\sqrt{w(f_n)\Delta f}$ and wbere

$$\omega_n = 2\pi f_n, \qquad f_n = n\Delta f$$

The second one is

$$I(t) = \sum_{n=1}^{N} c_n \cos (\omega_n t - \varphi_n) \qquad (2.8\text{–}6)$$

where φ_n is a random pbase angle distributed uniformly over the range $(0, 2\pi)$ and

$$c_n = [2w(f_n)\Delta f]^{1/2}$$

At an appropriate point in the analysis N and Δf are made to approach infinity and zero, respectively, in such a manner tbat the entire frequency band is covered by the summations (which then become integrations).

6. The normal distribution in several variables and the central limit theorem are discussed in sections 2.9 and 2.10.

Part III—Statistical Properties of Noise Current

1. The noise current is distributed normally. This has already been discussed in section 1.6 for the shot-effect. It is discussed again in section 3.1 using the concepts introduced in Part II, and the assumption, used tbroughout Part III, that the average value of the noise current $I(t)$ is zero. The probability that $I(t)$ lies between I and $I + dI$ is

$$\frac{dI}{\sqrt{2\pi\psi_0}} e^{-I^2/2\psi_0} \qquad (3.1\text{–}3)$$

where ψ_0 is the value of the correlation function, $\psi(\tau)$, of $I(t)$ at $\tau = 0$

$$\psi_0 = \psi(0) = \int_0^\infty w(f)\, df, \qquad (3.1\text{–}2)$$

$w(f)$ being the power spectrum of $I(t)$. ψ_0 is the mean square value of $I(t)$, i.e., the r.m.s. value of $I(t)$ is $\psi_0^{1/2}$.

The characteristic function (ch. f.) of this distribution is

$$\text{ave. } e^{iuI(t)} = \exp - \frac{\psi_0}{2} u^2 \qquad (3.1\text{–}6)$$

2. The probability that $I(t)$ lies between I_1 and $I_1 + dI$, and $I(t + \tau)$ lies between I_2 and $I_2 + dI_2$ when t is chosen at random is

$$[\psi_0^2 - \psi_\tau^2]^{-1/2} \frac{dI_1\, dI_2}{2\pi} \exp\left[\frac{-\psi_0 I_1^2 - \psi_0 I_2^2 + 2\psi_\tau I_1 I_2}{2(\psi_0^2 - \psi_\tau^2)}\right] \quad (3.2\text{–}4)$$

where ψ_τ is the correlation function $\psi(\tau)$ of $I(t)$:

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau \, df \quad (3.2\text{–}3)$$

The ch. f. for this distribution is

$$\text{ave. } e^{iuI(t)+ivI(t+\tau)} = \exp\left[-\frac{\psi_0}{2}(u^2 + v^2) - \psi_\tau uv\right] \quad (3.2\text{–}7)$$

3. The expected number of zeros per second of $I(t)$ is

$$\frac{1}{\pi}\left[-\frac{\psi''(0)}{\psi(0)}\right]^{1/2} = 2\left[\frac{\int_0^\infty f^2 w(f)\, df}{\int_0^\infty w(f)\, df}\right]^{1/2} \quad (3.3\text{–}11)$$

assuming convergence of the integrals. The primes denote differentiation with respect to τ:

$$\psi''(\tau) = \frac{d^2}{d\tau^2}\psi(\tau).$$

For an ideal band-pass filter whose pass band extends from f_a to f_b the expected number of zeros per second is

$$2\left[\frac{1}{3}\frac{f_b^3 - f_a^3}{f_b - f_a}\right]^{1/2} \quad (3.3\text{–}12)$$

When f_a is zero this becomes $1.155\, f_b$ and when f_a is very nearly equal to f_b it approaches $f_b + f_a$.

4. The problem of determining the distribution function for the length of the interval between two successive zeros of $I(t)$ seems to be quite difficult. In section 3.4 some related results are given which lead, in some circumstances, to approximations to the distribution. For example, for an ideal narrow band-pass filter the probability that the distance between two successive zeros lies between τ and $\tau + d\tau$ is approximately

$$\frac{d\tau}{2} \frac{a}{[1 + a^2(\tau - \tau_1)^2]^{3/2}}$$

where

$$a = \sqrt{3}\,\frac{(f_b + f_a)^2}{f_b - f_a}, \qquad \tau_1 = \frac{1}{f_b + f_a}$$

f_b and f_a being the upper and lower cut-off frequencies.

5. In section 3.5 several multiple integrals which occur in the work of Part III are discussed.

6. The distribution of the maxima of $I(t)$ is discussed in section 3.6. The expected number of maxima per second is

$$\frac{1}{2\pi}\left[-\frac{\psi_0^{(4)}}{\psi_0''}\right]^{1/2} = \left[\frac{\int_0^\infty f^4 w(f)\,df}{\int_0^\infty f^2 w(f)\,df}\right]^{1/2} \tag{3.6-6}$$

For a band-pass filter the expected number of maxima per second is

$$\left[\frac{3}{5}\,\frac{f_b^5 - f_a^5}{f_b^3 - f_a^3}\right]^{1/2} \tag{3.6-7}$$

For a low-pass filter where $f_a = 0$ this number is $0.775 f_b$.

The expected number of maxima per second lying above the line $I(t) = I_1$ is approximately, when I_1 is large,

$$e^{-I_1^2/2\psi_0} \times \tfrac{1}{2}[\text{the expected number of zeros of } I \text{ per second}] \tag{3.6-11}$$

where ψ_0 is the mean square value of $I(t)$.

For a low-pass filter the probability that a maximum chosen at random from the universe of maxima lies between I and $I + dI$ is approximately, when I is large,

$$\frac{\sqrt{5}}{3}\,y e^{-y^2/2}\,\frac{dI}{\psi_0^{1/2}} \tag{3.6-9}$$

where

$$y = \frac{I}{\psi_0^{1/2}}$$

7. When we pass noise through a relatively narrow band-pass filter one of the most noticeable features of an oscillogram of the output current is its fluctuating envelope. In sections 3.7 and 3.8 some statistical properties of this envelope, denoted by R or $R(t)$, are derived.

The probability that the envelope lies between R and $R + dR$ is

$$\frac{R}{\psi_0}\,e^{-R^2/2\psi_0}\,dR \tag{3.7-10}$$

where ψ_0 is the mean square value of $I(t)$. The probability that $R(t)$ lies between R_1 and $R_1 + dR_1$ and at the same time $R(t + \tau)$ lies between R_2 and $R_2 + dR_2$ when t is chosen at random is obtained by multiplying (3.7–13) by $dR_1\, dR_2$. For an ideal band-pass filter, the expected number of maxima of the envelope in one second is

$$.64110(f_b - f_a) \tag{3.8–15}$$

When R is large, say $y > 2.5$ where

$$y = \frac{R}{\psi_0^{1/2}}, \qquad \psi_0^{1/2} = \text{r.m.s. value of } I(t),$$

the probability that a maximum of the envelope, selected at random from the universe of such maxima, lies between R and $R + dR$ is approximately

$$1.13(y^2 - 1)e^{-y^2/2} \frac{dR}{\psi_0^{1/2}}$$

A curve for the corresponding probability density is shown for the range $0 \leq y \leq 4$. Curves which compare the distribution function of the maxima of R with other distribution functions of the same type are also given.

8. In section 3.9 some information is given regarding the statistical behavior of the random variable:

$$E = \int_{t_1}^{t_1+T} I^2(t)\, dt \tag{3.9–1}$$

where t_1 is chosen at random and $I(t)$ is a noise current with the power spectrum $w(f)$ and the correlation function $\psi(\tau)$. The average value m_T of E is $T\psi_0$ and its standard deviation σ_T is given by (3.9–9). For a relatively narrow band-pass filter

$$\frac{\sigma_T}{m_T} \sim \frac{1}{\sqrt{T(f_b - f_a)}}$$

when $T(f_b - f_a) \gg 1$. This follows from equation (3.9–10). An expression which is believed to approximate the distribution of E is given by (3.9–20).

9. In section 3.10 the distribution of a noise current plus one or more sinusoidal currents is discussed. For example, if I consists of two sine waves plus noise:

$$I = P \cos pt + Q \cos qt + I_N; \tag{3.10–20}$$

where p and q are incommensurable and the r.m.s. value of the noise current I_N is $\psi_0^{1/2}$, the probability density of the envelope R is

$$R \int_0^\infty r J_0(Rr) J_0(Pr) J_0(Qr) e^{-\psi_0 r^2/2}\, dr \tag{3.10–21}$$

where $J_0(\ )$ is a Bessel function.

Curves showing the probability density and distribution function of R, when $Q = 0$, for various ratios of P/r.m.s. I_N are given.

10. In section 3.11 it is pointed out that the representations (2.8–1) and (2.8–6) of the noise current as the sum of a great number of sinusoidal components are not the only ones which may be used in deriving the results given in the preceding sections of Part III. The sbot effect representation

$$I(t) = \sum_{-\infty}^{+\infty} F(t - t_k)$$

studied in Part I may also be used.

Part IV—Noise Through Non-Linear Devices

1. Suppose that the power spectrum of the voltage V applied to tbe square-law device

$$I = \alpha V^2 \tag{4.1–1}$$

is confined to a relatively narrow band. The total low-frequency output current $I_{\ell\ell}$ may be expressed as the sum

$$I_{\ell\ell} = I_{dc} + I_{\ell_f} \tag{4.1–2}$$

where I_{dc} is the d.c. component and I_{ℓ_f} is the variable component. When none of the low-frequency band is eliminated (by audio frequency filters)

$$I_{\ell\ell} = \frac{\alpha R^2}{2} \tag{4.1–6}$$

wbere R is the envelope of V. If V is of the form

$$V = V_N + P \cos pt + Q \cos qt, \tag{4.1–4}$$

where V_N is a noise voltage wbose mean square value is ψ_0, then

$$I_{dc} = \alpha\left(\psi_0 + \frac{P^2}{2} + \frac{Q^2}{2}\right)$$

$$\overline{I_{\ell_f}^2} = \alpha^2\left[\psi_0^2 + P^2\psi_0 + Q^2\psi_0 + \frac{P^2Q^2}{2}\right] \tag{4.1–16}$$

2. If instead of a square-law device we have a linear rectifier,

$$I = \begin{cases} 0 & V < 0 \\ \alpha V, & V > 0 \end{cases} \tag{4.2–1}$$

the total low-frequency output is

$$I_{\ell\ell} = \frac{\alpha R}{\pi} \tag{4.2–2}$$

When V is a sine wave plus noise, $V_N + P \cos pt$,

$$I_{dc} = \alpha \left(\frac{\psi_0}{2\pi} \right)^{1/2} {}_1F_1(-\tfrac{1}{2}; 1; -x) \tag{4.2-3}$$

$$\overline{I_{lf}^2} = \frac{\alpha^2}{\pi^2} (P^2 + 2\psi_0) \tag{4.2-6}$$

where ${}_1F_1$ is a hypergeometric function and

$$x = \frac{P^2}{2\psi_0} = \frac{\text{Ave. sine wave power}}{\text{Ave. noise power}} \tag{4.2-4}$$

When x is large

$$\overline{I_{lf}^2} \sim \frac{\alpha^2 \psi_0}{\pi^2} \left[1 - \frac{1}{4x} \cdots \right] \tag{4.2-7}$$

If V consists of two sine waves plus noise, I_{dc} consists of a hypergeometric function of two variables. The equations running from (4.2-9) to (4.2-15) are concerned with this case. About the only simple equation is

$$\overline{I_{lf}^2} = \frac{\alpha^2}{\pi^2} [2\psi_0 + P^2 + Q^2] \tag{4.2-14}$$

3. The expressions (4.1-6) and (4.2-2) for I_{lf} in terms of the envelope R of V, namely

$$\frac{\alpha R^2}{2} \quad \text{and} \quad \frac{\alpha R}{\pi},$$

are special cases of a more general result

$$I_{lf} = A_0(R) = \frac{1}{2\pi} \int_C F(iu) J_0(uR)\, du. \tag{4.3-11}$$

In this expression $J_0(uR)$ is a Bessel function. The path of integration C and the function $F(iu)$ are chosen so that the relation between I and V may be expressed as

$$I = \frac{1}{2\pi} \int_C F(iu) e^{iVu}\, du. \tag{4A-1}$$

A table giving $F(iu)$ and C for a number of common non-linear devices is shown in Appendix 4A.

If this relation is used to study the biased linear rectifier.

$$I = \begin{cases} 0, & V < B \\ V - B, & V > B \end{cases}$$

for the case in which V is $V_N + P \cos pt$, we find

$$I_{dc} \sim -\frac{B}{2} + \frac{P}{\pi} + \frac{B^2 + \psi_0}{2\pi P}$$
$$\overline{I_{lf}^2} \sim \frac{P^2 - B^2}{\pi^2 P^2}\,\psi_0 \tag{4.3–17}$$

when $P \gg |B|$, $P^2 \gg \psi_0$ where ψ_0 is the mean square value of V_N.

4. When V is confined to a relatively narrow band and there are no audio-frequency filters, the probability density and all the associated statistical properties of $I_{\ell f}$ may be obtained by expressing $I_{\ell f}$ as a function of the envelope R of V and then using the probability density of R. When V is $V_N + P \cos pt + Q \cos qt$ this probability density is given by the integral, (3.10–21) (which is the integral containing three Bessel functions stated in the above summary of Part III). When V consists of three sine waves plus noise there are four J_0's in the integrand, and so on. Expressions for $\overline{R^n}$ when R has the above distribution are given by equations (3.10–25) and (3.10–27).

When audio-frequency filters remove part of the low-frequency band the statistical properties, except the mean square value, of the resulting current are hard to compute. In section 4.3 it is shown that as the output band is chosen narrower and narrower, the statistical properties of the output current approach those of a random noise current.

5. The sections in Part IV from 4.4 onward are concerned with the problem: Given a non-linear device and an input voltage consisting of noise alone or of a signal plus noise. What is tbe power spectrum of the output? A survey of the methods available for the solution of this problem is given in section 4.4.

6. When a noise voltage V_N with the power spectrum $w(f)$ is applied to the square-law device

$$I = \alpha V^2 \tag{4.1–1}$$

the power spectrum of the output current I is, when $f \neq 0$,

$$W(f) = \alpha^2 \int_{-\infty}^{+\infty} w(x)w(f - x)\,dx \tag{4.5–5}$$

where $w(-x)$ is defined to equal $w(x)$. The power spectrum of I when V is either $P \cos pt + V_N$ or

$$Q(1 + k \cos pt) \cos qt + V_N$$

is considered in the portion of section 4.5 containing equations (4.5–10) to (4.5–17).

7. A method discovered independently by Van Vleck and North shows that the correlation function $\Psi(\tau)$ of the output current for an unbiased linear rectifier is

$$\Psi(\tau) = \frac{\psi_\tau}{4} + \frac{\psi_0}{2} {}_2F_1\left[-\tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}; \frac{\psi_\tau^2}{\psi_0^2}\right] \tag{4.7–6}$$

where the input voltage is V_N. The correlation function $\psi(\tau)$ of V_N is denoted by ψ_τ and the mean square value of V_N is ψ_0. The power spectrum $W(f)$ of I may be obtained from

$$W(f) = 4 \int_0^\infty \Psi(\tau) \cos 2\pi f\tau \, d\tau \tag{4.6–1}$$

by expanding the hypergeometric function and integrating termwise using

$$G_n(f) = \int_0^\infty \psi_\tau^n \cos 2\pi f\tau \, d\tau. \tag{4C–1}$$

Appendix 4C is devoted to the problem of evaluating the integral for $G_n(f)$.

8. Another method of obtaining the correlation function $\psi(\tau)$ of I, termed the "characteristic function method," is explained in section 4.8. It is illustrated in section 4.9 where formulas for $\Psi(\tau)$ and $W(f)$ are developed when the voltage $P \cos pt + V_N$ is applied to a general non-linear device.

9. Several miscellaneous results are given in section 4.10. The characteristic function method is used to obtain the correlation function for a square-law device. The general formulas of section 4.9 are applied to the case of a ν^{th} law rectifier when the input noise spectrum has a normal law distribution. Some remarks are also made concerning the audio-frequency output of a linear rectifier when the input voltage V is

$$Q(1 + r \cos pt) \cos qt + V_N.$$

10. A discussion of the hypergeometric function ${}_1F_1(a; c; x)$, which often occurs in problems concerning a sine wave plus noise, is given in Appendix 4B.

PART I

THE SHOT EFFECT

The shot effect in vacuum tubes is a typical example of noise. It is due to fluctuations in the intensity of the stream of electrons flowing from the cathode to the anode. Here we analyze a simplified form of the shot effect.

1.1 The Probability of Exactly K Electrons Arriving at the Anode in Time T

The fluctuations in the electron stream are supposed to be random. We shall treat this randomness as follows. We count the number of electrons flowing in a long interval of time T measured in seconds. Suppose there are K_1. Repeating this counting process for many intervals all of length T gives a set of numbers $K_2, K_3 \cdots K_M$ where M is the total number of intervals. The average number ν, of electrons per second is defined as

$$\nu = \lim_{M\to\infty} \frac{K_1 + K_2 + \cdots + K_M}{MT} \tag{1.1-1}$$

where we assume that this limit exists. As M is increased with T being held fixed some of the K's will have the same value. In fact, as M increases the number of K's having any particular value will tend to increase. This of course is based on the assumption that the electron stream is a steady flow upon which random fluctuations are superposed. The probability of getting K electrons in a given trial is defined as

$$p(K) = \lim_{M\to\infty} \frac{\text{Number of trials giving exactly } K \text{ electrons}}{M} \tag{1.1-2}$$

Of course $p(K)$ also depends upon T. We assume that the randomness of the electron stream is such than the probability that an electron will arrive at the anode in the interval $(t, t + \Delta t)$ is $\nu\Delta t$ where Δt is such that $\nu\Delta t \ll 1$, and that this probability is independent of what has happened before time t or will happen after time $t + \Delta t$.

This assumption is sufficient to determine the expression for $p(K)$ which is

$$p(K) = \frac{(\nu T)^K}{K!} e^{-\nu T} \tag{1.1-3}$$

This is the "law of small probabilities" given by Poisson. One method of derivation sometimes used can be readily illustrated for the case $K = 0$. Thus, divide the interval, $(0, T)$ into M intervals each of length $\Delta t = \frac{T}{M}$. Δt is taken so small that $\nu\Delta t$ is much less than unity. (This is the "small probability" that an electron will arrive in the interval Δt). The probability that an electron will not arrive in the first sub-interval is $(1 - \nu\Delta t)$. The probability that one will not arrive in either the first or the second sub-interval is $(1 - \nu\Delta t)^2$. The probability that an electron will not arrive in any of the M intervals is $(1 - \nu\Delta t)^M$. Replacing M by $T/\Delta t$ and letting $\Delta t \to 0$ gives

$$p(0) = e^{-\nu T}$$

The expressions for $p(1)$, $p(2)$, $\cdots$ $p(K)$ may be derived in a somewhat similar fashion.

1.2 Statement of Campbell's Theorem

Suppose that the arrival of an electron at the anode at time $t = 0$ produces an effect $F(t)$ at some point in the output circuit. If the output circuit is such that the effects of the various electrons add linearly, the total effect at time t due to all the electrons is

$$I(t) = \sum_{k=-\infty}^{+\infty} F(t - t_k) \tag{1.2-1}$$

where the k^{th} electron arrives at t_k and the series is assumed to converge.

Campbell's theorem[2] states that the average value of $I(t)$ is

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.2-2}$$

and the mean square value of the fluctuation about this average is

$$\overline{(I(t) - \overline{I(t)})^2} = \nu \int_{-\infty}^{+\infty} F^2(t)\, dt \tag{1.2-3}$$

where ν is the average number of electrons arriving per second.

The statement of the theorem is not precise until we define what we mean by "average". From the form of the equations the reader might be tempted to think of a time average; e.g. the value

$$\operatorname*{Lim}_{T\to\infty} \frac{1}{T} \int_0^T I(t)\, dt \tag{1.2-4}$$

However, in the proof of the theorem the average is generally taken over a great many intervals of length T with t held constant. The process is somewhat similar to that employed in (1.1) and in order to make it clear we take the case of $\overline{I(t)}$ for illustration. We observe $I(t)$ for many, say M, intervals each of length T where T is large in comparison with the interval over which the effect $F(t)$ of the arrival of a single electron is appreciable. Let ${}_nI(t')$ be the value of $I(t)$, t' seconds after the beginning of the n^{th} interval. t' is equal to t plus a constant depending upon the beginning time of the interval. We put the subscript in front because we wish to reserve the usual place for another subscript later on. The value of $\overline{I(t')}$ is then defined as

$$\overline{I(t')} = \operatorname*{Limit}_{M\to\infty} \frac{1}{M} [{}_1I(t') + {}_2I(t') + \cdots + {}_MI(t')] \tag{1.2-5}$$

and this limit is assumed to exist. The mean square value of the fluctuation of $I(t')$ is defined in much the same way.

[2] *Proc. Camb. Phil. Soc.* 15 (1909), 117–136, 310–328. Our proof is similar to one given by J. M. Whittaker, *Proc. Camb. Phil. Soc.* 33 (1937), 451–458.

Actually, as the equations (1.2–2) and (1.2–3) of Campbell's theorem show, these averages and all the similar averages encountered later turn out to be independent of the time. When this is true and when the M intervals in (1.2–5) are taken consecutively the time average (1.2–4) and the average (1.2–5) become the same. To show this we multiply both sides of (1.2–5) by dt' and integrate from 0 to T:

$$\begin{aligned}\overline{I(t')} &= \operatorname*{Limit}_{M\to\infty} \frac{1}{MT} \sum_{m=1}^{M} \int_0^T {}_mI(t')\, dt' \\ &= \operatorname*{Limit}_{M\to\infty} \frac{1}{MT} \int_0^{MT} I(t)\, dt\end{aligned} \tag{1.2–6}$$

and this is the same as the time average (1.2–4) if the latter limit exists.

1.3 Proof of Campbell's Theorem

Consider the case in which exactly K electrons arrive at the anode in an interval of length T. Before the interval starts, we think of these K electrons as fated to arrive in the interval $(0, T)$ but any particular electron is just as likely to arrive at one time as any other time. We shall number these fated electrons from one to K for purposes of identification but it is to be emphasized that the numbering has nothing to do with the order of arrival. Thus, if t_k be the time of arrival of electron number k, the probability that t_k lies in the interval $(t, t + dt)$ is dt/T.

We take T to be very large compared with the range of values of t for which $F(t)$ is appreciably different from zero. In physical applications such a range usually exists and we shall call it Δ even though it is not very definite. Then, when exactly K electrons arrive in the interval $(0, T)$ the effect is approximately

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.3–1}$$

the degree of approximation being very good over all of the interval except within Δ of the end points.

Suppose we examine a large number M of intervals of length T. The number having exactly K arrivals will be, to a first approximation $M\, p(K)$ where $p(K)$ is given by (1.1–3). For a fixed value of t and for each interval having K arrivals, $I_K(t)$ will have a definite value. As $M \to \infty$, the average value of the $I_K(t)$'s, obtained by averaging over the intervals, is

$$\begin{aligned}\overline{I_K(t)} &= \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T} \sum_{k=1}^{K} F(t - t_k) \\ &= \sum_{k=1}^{K} \int_0^T \frac{dt_k}{T} F(t - t_k)\end{aligned} \tag{1.3–2}$$

and if $\Delta < t < T - \Delta$, we have effectively

$$\overline{I_K(t)} = \frac{K}{T} \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.3-3}$$

If we now average $I(t)$ over all of the M intervals instead of only over those having K arrivals, we get, as $M \to \infty$,

$$\begin{aligned} \overline{I(t)} &= \sum_{K=0}^{\infty} p(K)\overline{I_K(t)} \\ &= \sum_{K=0}^{\infty} \frac{K}{T} \frac{(\nu T)^K}{K!} e^{-\nu T} \int_{-\infty}^{+\infty} F(t)\, dt \\ &= \nu \int_{-\infty}^{+\infty} F(t)\, dt \end{aligned} \tag{1.3-4}$$

and this proves the first part of the theorem. We have used this rather elaborate proof to prove the relatively simple (1.3–4) in order to illustrate a method which may be used to prove more complicated results. Of course, (1.3–4) could be established by noting that the integral is the average value of the effect produced by one arrival, the average being taken over one second, and that ν is the average number of arrivals per second.

In order to prove the second part, (1.2–3) of Campbell's theorem we first compute $\overline{I^2(t)}$ and use

$$\begin{aligned} \overline{(I(t) - \overline{I(t)})^2} &= \overline{I^2(t)} - 2\, \overline{I(t)\overline{I(t)}} + \overline{I(t)}^2 \\ &= \overline{I^2(t)} - \overline{I(t)}^2 \end{aligned} \tag{1.3-5}$$

From the definition (1.3–1) of $I_K(t)$,

$$I_K^2(t) = \sum_{k=1}^{K} \sum_{m=1}^{K} F(t - t_k)F(t - t_m)$$

Averaging this over all values of $t_1, t_2, \cdots t_K$ with t beld fixed as in (1.3–2),

$$\overline{I_K^2(t)} = \sum_{k=1}^{K} \sum_{m=1}^{K} \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T} F(t - t_k)F(t - t_m)$$

Tbe multiple integral has two different values. If $k = m$ its value is

$$\int_0^T F^2(t - t_k) \frac{dt_k}{T}$$

and if $k \neq m$ its value is

$$\int_0^T F(t - t_k) \frac{dt_k}{T} \int_0^T F(t - t_m) \frac{dt_m}{T}$$

Counting up the number of terms in the double sum shows that there are K of them having the first value and $K^2 - K$ baving the second value. Hence, if $\Delta < t < T - \Delta$ we have

$$\overline{I_K^2(t)} = \frac{K}{T} \int_{-\infty}^{+\infty} F^2(t)\, dt + \frac{K(K-1)}{T^2} \left[\int_{-\infty}^{+\infty} F(t)\, dt \right]^2$$

Averaging over all the intervals instead of only those having K arrivals gives

$$\overline{I^2(t)} = \sum_{K=0}^{\infty} p(K)\, \overline{I_K^2(t)}$$

$$= \nu \int_{-\infty}^{+\infty} F^2(t)\, dt + \overline{I(t)}^2$$

where the summation with respect to K is performed as in (1.3–4), and after summation the value (1.3–4) for $\overline{I(t)}$ is used. Comparison with (1.3–5) establishes the second part of Campbell's theorem.

1.4 The Distribution of $I(t)$

When certain conditions are satisfied the proportion of time which $I(t)$ spends in the range $I, I + dI$ is $P(I)dI$ where, as $\nu \to \infty$, the probability density $P(I)$ approaches

$$\frac{1}{\sigma_I \sqrt{2\pi}} e^{-(I-\bar{I})^2/2\sigma_I^2} \qquad (1.4\text{–}1)$$

where $\bar{I}$ is the average of $I(t)$ given by (1.2–2) and the square of the standard deviation σ_I, i.e. the variance of $I(t)$, is given by (1.2–3). This normal distribution is the one which would be expected by virtue of the "central limit theorem" in probability. This states that, under suitable conditions, the distribution of the sum of a large number of random variables tends toward a normal distribution whose variance is the sum of the variances of the individual variables. Similarly the average of the normal distribution is the sum of the averages of the individual variables.

So far, we have been speaking of the limiting form of the probability density $P(I)$. It is possible to write down an explicit expression for $P(I)$, which, however, is quite involved. From this expression the limiting form may be obtained. We now obtain this expression. In line with the discussion given of Campbell's theorem, we seek the probability density $P(I)$ of the values of $I(t)$ observed at t seconds from the beginning of eacb of a large number, M, of intervals, each of length T.

Prohability that $I(t)$ lies in range $(I, I + dI)$

$$= \sum_{K=0}^{\infty} \text{(Prohability of exactly } K \text{ arrivals)} \times \text{(Prohability that if there are exactly } K \text{ arrivals, } I_K(t) \text{ lies in } (I, I + dI)).$$

Denoting the last prohability in the summation hy $P_K(I)dI$, using notation introduced earlier, and cancelling out the factor dI gives

$$P(I) = \sum_{K=0}^{\infty} p(K)P_K(I) \tag{1.4–2}$$

We shall compute $P_K(I)$ by the method of "characteristic functions"[3] from the definition

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.3–1}$$

of $I_K(t)$. The method will he used in its simplest form: the prohahility that the sum

$$x_1 + x_2 + \cdots + x_K$$

of K independent random variahles lies hetween X and $X + dX$ is

$$dX \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iXu} \prod_{k=1}^{K} (\text{average value of } e^{ix_k u})\, du \tag{1.4–3}$$

The average value of $e^{ix_k u}$, i.e., the characteristic function of the distrihution of x_k, is ohtained by averaging over the values of x_k. Although this is the simplest form of the method it is also the least general in that the integral does not converge for some important cases. The distrihution which gives a prohability of $\frac{1}{2}$ that $x_k = -1$ and $\frac{1}{2}$ that $x_k = +1$ is an example of such a case. However, we may still use (1.4–3) formally in such cases hy employing the relation

$$\int_{-\infty}^{+\infty} e^{-iau}\, du = 2\pi\delta(a) \tag{1.4–4}$$

where $\delta(a)$ is zero except at $a = 0$ where it is infinite and its integral from $a = -\epsilon$ to $a = +\epsilon$ is unity where $\epsilon > 0$.

When we identify x_k with $F(t - t_k)$ we see that the average value of $e^{ix_k u}$ is

$$\frac{1}{T} \int_0^T \exp\,[iuF(t - t_k)]\, dt_k$$

[3] The essentials of this method are due to Laplace. A few remarks on its history are given by E. C. Molina, *Bull. Amer. Math. Soc.*, 36 (1930), pp. 369–392. An account of the method may be found in any one of several texts on probability theory. We mention "Random Variables and Probability Distributions," by H. Cramér, Camb. Tract in Math. and Math. Phys. No. 36 (1937), Chap. IV. Also "Introduction to Mathematical Probability," by J. V. Uspensky, McGraw-Hill (1937), pages 240, 264, and 271–278.

All of the K characteristic functions are the same and hence, from (1.4–3), $P_K(I)dI$ is

$$dI \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iIu} \left(\frac{1}{T} \int_0^T \exp\left[iuF(t-\tau)\right] d\tau \right)^K du$$

Although in deriving this relation we have taken $K > 0$, it also holds for $K = 0$ (provided we use (1.4–4)). In this case $P_0(I) = \delta(I)$, because $I = 0$ when no electrons arrive.

Inserting our expression for $P_K(I)$ and the expression (1.1–3) for $p(K)$ in (1.4–2) and performing the summation gives

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-iIu - \nu T + \nu \int_0^T \exp\left[iuF(t-\tau)\right] d\tau \right) du \qquad (1.4\text{–}5)$$

The first exponential may be simplified somewhat. Using

$$\nu T = \nu \int_0^T d\tau$$

permits us to write

$$-\nu T + \nu \int_0^T \exp\left[iuF(t-\tau)\right] d\tau = \nu \int_0^T \left(\exp\left[iuF(t-\tau)\right] - 1\right) d\tau$$

Suppose that $\Delta < t < T - \Delta$ where Δ is the range discussed in connection with equation (1.3–1). Taking $|F(t-\tau)| = 0$ for $|t-\tau| > \Delta$ then enables us to write the last expression as

$$\nu \int_{-\infty}^{+\infty} \left[e^{iuF(t)} - 1\right] dt \qquad (1.4\text{–}6)$$

Placing this in (1.4–5) yields the required expression for $P(I)$:

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-iIu + \nu \int_{-\infty}^{+\infty} \left[e^{iuF(t)} - 1\right] dt \right) du \qquad (1.4\text{–}7)$$

An idea of the conditions under which the normal law (1.4–1) is approached may be obtained from (1.4–7) by expanding (1.4–6) in powers of u and determining when the terms involving u^3 and higher powers of u may be neglected. This is taken up for a slightly more general form of current in section 1.6.

1.5 Extension of Campbell's Theorem

In section 1.2 we have stated Campbell's theorem. Here we shall give an extension of it. In place of the expression (1.2–1) for the $I(t)$ of the shot effect we shall deal with the current

$$I(t) = \sum_{k=-\infty}^{+\infty} a_k F(t - t_k) \tag{1.5–1}$$

where $F(t)$ is the same sort of function as before and where $\cdots a_1, a_2, \cdots a_k, \cdots$ are independent random variables all having the same distribution. It is assumed that all of the moments $\overline{a^n}$ exist, and that the events occur at random

The extension states that the nth semi-invariant of the probability density $P(I)$ of I, where I is given by (1.5–1), is

$$\lambda_n = \nu \overline{a^n} \int_{-\infty}^{+\infty} [F(t)]^n \, dt \tag{1.5–2}$$

where ν is the expected number of events per second. The semi-invariants of a distribution are defined as the coefficients in the expansion

$$\log_e (\text{ave. } e^{iIu}) = \sum_{n=1}^{N} \frac{\lambda_n}{n!} (iu)^n + o(u^N) \tag{1.5–3}$$

i.e. as the coefficients in the expansion of the logarithm of the characteristic function. The λ's are related to the moments of the distribution. Thus if $m_1, m_2, \cdots$ denote the first, second $\cdots$ moments about zero we have

$$\text{ave. } e^{iIu} = 1 + \sum_{n=1}^{N} \frac{m_n}{n!} (iu)^n + o(u^N)$$

By combining this relation with the one defining the λ's it may be shown that

$$\begin{aligned} \bar{I} &= m_1 = \lambda_1 \\ \overline{I^2} &= m_2 = \lambda_2 + \lambda_1 m_1 \\ \overline{I^3} &= m_3 = \lambda_3 + 2\lambda_2 m_1 + \lambda_1 m_2 \\ &\cdots\cdots\cdots \end{aligned}$$

It follows that $\lambda_1 = \bar{I}$ and $\lambda_2 = \text{ave. } (I - \bar{I})^2$. Hence (1.5–2) yields the original statement of Campbell's theorem when we set n equal to one and two and also take all the a's to be unity.

The extension follows almost at once from the generalization of expression (1.4–7) for the probability density $P(I)$. By proceeding as in section 1.4 and identifying x_k with $a_k F(t - t_k)$ we see that

$$\text{ave. } e^{ix_k u} = \frac{1}{T} \int_{-\infty}^{+\infty} q(a) \, da \int_0^T \exp [iuaF(t - t_k)] \, dt_k$$

where $q(a)$ is the probability density function for the a's. It turns out that the probability density $P(I)$ of I as defined by (1.5–1) is

$$P(I) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-iIu + \nu \int_{-\infty}^{+\infty} q(a)\, da \int_{-\infty}^{+\infty} [e^{iuaF(t)} - 1]\, dt\right) du \quad (1.5\text{–}4)$$

The logarithm of the characteristic function of $P(I)$ is, from (1.5–4),

$$\nu \int_{-\infty}^{+\infty} q(a)\, da \int_{-\infty}^{+\infty} [e^{iuaF(t)} - 1]\, dt = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \nu \int_{-\infty}^{+\infty} q(a)\, da\, a^n \int_{-\infty}^{+\infty} F^n(t)\, dt$$

Comparison with tbe series (1.5–3) defining the semi-invariants gives the extension of Campbell's theorem stated by (1.5–2).

Other extensions of Campbell's theorem may be made. For example, suppose in the expression (1.5–1) for $I(t)$ that $t_1, t_2, \cdots t_k, \cdots$ while still random variables, are no longer necessarily distributed according to the laws assumed above. Suppose now that the prohability density $p(x)$ is given where x is the interval between two successive events:

$$t_2 = t_1 + x_1 \quad (1.5\text{–}5)$$

$$t_3 = t_2 + x_2 = t_1 + x_1 + x_2$$

and so on. For the case treated above

$$p(x) = \nu e^{-\nu x}. \quad (1.5\text{–}6)$$

We assume that the expected number of events per second is still ν.

Also we take the special, but important, case for which

$$F(t) = 0, \qquad t < 0 \quad (1.5\text{–}7)$$

$$F(t) = e^{-\alpha t}, \qquad t > 0.$$

For a very long interval extending from $t = t_1$ to $t = T + t_1$ inside of which there are exactly K events we have, if t is not near the ends of the interval,

$$I(t) = a_1F(t - t_1) + a_2F(t - t_1 - x_1) + \cdots + a_{K+1}F(t - t_1 - x_1 \cdots - x_K)$$

$$= a_1F(t') + a_2F(t' - x_1) + \cdots + a_{K+1}F(t' - x_1 - \cdots - x_K)$$

where $q(a)$ is the probability density function for the a's. It turns out that the probability density $P(I)$ of I as defined by (1.5–1) is

$$P(I) = \frac{1}{2\pi} \int_{\infty}^{+\infty} \exp\left(-iIu + \nu \int_{\infty}^{+\infty} q(a)\, da \int_{\infty}^{+\infty} [e^{iuaF(t)} - 1]\, dt\right) du \quad (1.5\text{–}4)$$

The logarithm of the characteristic function of $P(I)$ is, from (1.5–4),

$$\nu \int_{\infty}^{+\infty} q(a)\, da \int_{\infty}^{+\infty} [e^{iuaF(t)} - 1]\, dt = \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} \nu \int_{\infty}^{+\infty} q(a)\, da\, a^n \int_{\infty}^{+\infty} F^n(t)\, dt$$

Comparison with the series (1.5–3) defining the semi-invariants gives the extension of Campbell's theorem stated by (1.5–2).

Other extensions of Campbell's theorem may be made. For example, suppose in the expression (1.5–1) for $I(t)$ that $t_1, t_2, \cdots t_k, \cdots$ while still random variables, are no longer necessarily distributed according to the laws assumed above. Suppose now that the probability density $p(x)$ is given where x is the interval between two successive events:

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For a very long interval extending from $t = t_1$ to $t = T + t_1$ inside of which there are exactly K events we have, if t is not near the ends of the interval,

$$I(t) = a_1F(t - t_1) + a_2F(t - t_1 - x_1) + \cdots + a_{K+1}F(t - t_1 - x_1 \cdots - x_K)$$

$$= a_1F(t') + a_2F(t' - x_1) + \cdots + a_{K+1}F(t' - x_1 - \cdots - x_K)$$

$$I^2(t) = a_1^2F^2(t') + a_2^2F^2(t' - x_1) + \cdots + a_{K+1}^2F^2(t' - x_1 \cdots - x_K)$$
$$+ 2a_1a_2F(t')F(t' - x_1) + \cdots + 2a_1a_{K+1}F(t')F(t' - x_1 \cdots - x_K)$$
$$+ 2a_2a_3F(t' - x_1)F(t' - x_1 - x_2) + \cdots + \cdots$$

where $t' = t - t_1$. If we integrate $I^2(t)$ over the entire interval $0 < t' < T$ and drop the primes we get approximately

$$\int_0^T I^2(t)dt = (a_1^2 + \cdots + a_{K+1}^2)\varphi(0)$$
$$+ 2a_1a_2\varphi(x_1) + 2a_1a_3\varphi(x_1 + x_2) + \cdots + 2a_1a_{K+1}\varphi(x_1 + \cdots + x_K)$$
$$+ 2a_2a_3\varphi(x_2) + \cdots + \cdots + 2a_Ka_{K+1}\varphi(x_K)$$

where

$$\varphi(x) = \int_{-\infty}^{+\infty} F(t)F(t - x)\, dx$$

When we divide both sides by T and consider K and T to be very large,

$$\frac{K}{T}\frac{a_1^2 + \cdots + a_{K+1}^2}{K}\varphi(0) \approx \nu\overline{a^2}\varphi(0)$$

$$\frac{1}{T}[a_1 a_2 \varphi(x_1) + a_2 a_3 \varphi(x_2) + \cdots + a_K a_{K+1} \varphi(x_K)] = \frac{K}{T} \text{ average } a_k a_{k+1} \varphi(x_k)$$
$$\approx \nu\bar{a}^2 \int_0^\infty \varphi(x)p(x)\, dx$$

$$\frac{1}{T}[a_1 a_3 \varphi(x_1 + x_2) + \cdots] = \frac{K-1}{T} \text{ ave. } a_k a_{k+3} \varphi(x_k + x_{k+1})$$
$$\approx \nu\bar{a}^2 \int_0^\infty dx_1 \int_0^\infty dx_2\, p(x_1)p(x_2)\varphi(x_1 + x_2)$$

Consequently

$$\overline{I^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T I^2(t)\, dt$$
$$= \nu\overline{a^2}\varphi(0) + 2\nu\bar{a}^2\left[\int_0^\infty p(x)\varphi(x)\, dx \right.$$
$$\left. + \int_0^\infty dx_1 \int_0^\infty dx_2\, p(x_1)p(x_2)\varphi(x_1 + x_2) + \cdots\right]$$

For our special exponential form (1.5–7) for $F(t)$,

$$\varphi(x) = \frac{e^{-\alpha x}}{2\alpha}$$

and the multiple integrals occurring in the expression for $\overline{I^2(t)}$ may be written in terms of powers of

$$q = \int_0^\infty p(x)e^{-\alpha x}\,dx \qquad (1.5\text{–}8)$$

Thus

$$2\alpha\overline{I^2(t)} = \nu\overline{a^2} + 2\bar{a}^2\nu\frac{q}{1-q}$$

and since

$$\bar{I}(t) = \nu\bar{a}\int_{-\infty}^{+\infty} F(t)\,dt = \nu\bar{a}/\alpha$$

we have

$$\overline{I^2(t)} - \overline{I(t)}^2 = \frac{\nu\overline{a^2}}{2\alpha} + \left(\frac{\nu\bar{a}}{\alpha}\right)^2\left[\frac{\alpha q}{\nu(1-q)} - 1\right] \qquad (1.5\text{–}9)$$

Equations (1.5–8) and (1.5–9) give us an extension of Campbell's theorem subject to the restrictions discussed in connection with equations (1.5–5) and (1.5–7). Other generalizations have been made[4] but we shall leave the subject here. The reader may find it interesting to verify that (1.5–9) gives the correct answer when $p(x)$ is given by (1.5–6), and also to investigate the case when the events are spaced equally.

1.6 Approach of Distribution of I to a Normal Law

In section 1.5 we saw that the probability density $P(I)$ of the noise current I may be expressed formally as

$$P(I) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left[-iIu + \sum_{n=1}^{\infty}(iu)^n\lambda_n/n!\right]du \qquad (1.6\text{–}1)$$

where λ_n is the nth semi-invariant given by (1.5–2). By setting

$$\lambda_2 = \sigma^2$$

$$x = \frac{I - \lambda_1}{\sigma} = \frac{I - \bar{I}}{\sigma} \qquad (1.6\text{–}2)$$

[4] See E. N. Rowland, *Proc. Camb. Phil. Soc.* 32 (1936), 580–597. He extends the theorem to the case where there are two functions instead of a single one, which we here denote by $I(t)$. According to a review in the Zentralblatt für Math., 19, p. 224, Khintchine in the Bull. Acad. Sci. URSS, sér. Math. Nr. 3 (1938), 313–322, has continued and made precise the earlier work of Rowland.

expanding

$$\exp \sum_{n=3}^{\infty} (iu)^n \lambda_n / n!$$

as a power series in u, integrating termwise using

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (iu\sigma)^n \exp\left[-iu\sigma x - \frac{u^2\sigma^2}{2}\right] du = (-)^n \sigma^{-1} \varphi^{(n)}(x),$$

$$\varphi^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \frac{d^n}{dx^n} e^{-x^2/2}$$

and finally collecting terms according to their order in powers of $\nu^{-1/2}$, gives

$$P(I) \sim \sigma^{-1}\varphi^{(0)}(x) - \frac{\lambda_3 \sigma^{-4}}{3!}\varphi^{(3)}(x) + \left[\frac{\lambda_4 \sigma^{-5}}{4!}\varphi^{(4)}(x) + \frac{\lambda_3^2 \sigma^{-7}}{72}\varphi^{(6)}(x)\right] + \cdots \tag{1.6-3}$$

The first term is $0(\nu^{-1/2})$, the second term is $0(\nu^{-1})$, and the term within brackets is $0(\nu^{-3/2})$. This is Edgeworth's series.[5] The first term gives the normal distribution and the remaining terms show how this distribution is approached as $\nu \rightarrow \infty$.

1.7 The Fourier Components of $I(t)$

In some analytical work noise current is represented as

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T}\right) \tag{1.7-1}$$

where at a suitable place in the work T and N are allowed to become infinite. The coefficients a_n and b_n, $1 \leq n \leq N$, are regarded as independent random variables distributed about zero according to a normal law.

It appears that the association of (1.7–1) with a sequence of disturbances occurring at random goes back many years. Rayleigh[6] and Gouy suggested that black-body radiation and white light might both be regarded as sequences of irregularly distributed impulses.

Einstein[7] and von Laue have discussed the normal distribution of the coefficients in (1.7–1) when it is used to represent black-body radiation, this radiation being the resultant produced by a great many independent os-

[5] See, for example, pp. 86–87, in "Random Variables and Probability Distributions" by H. Cramér, *Cambridge Tract* No. 36 (1937).

[6] *Phil. Mag.* Ser. 5, Vol. 27 (1889) pp. 460-469.

[7] A. Einstein and L. Hopf, *Ann. d. Physik* 33 (1910) pp. 1095–1115.
M. V. Laue, *Ann. d. Physik* 47 (1915) pp. 853–878.
A. Einstein, *Ann. d. Physik* 47 (1915) pp. 879–885.
M. V. Laue, *Ann. d. Physik* 48 (1915) pp. 668–680.
I am indebted to Prof. Goudsmit for these references.

cillators. Some argument arose as to whether the coefficients in (1.7–1) were statistically independent or not. It was finally decided that they are independent.

The shot effect current has been represented in this way by Schottky.[8] The Fourier series representation has been discussed by H. Nyquist[9] and also by Goudsmit and Weiss. Remarks made by A. Schuster[10] are equivalent to the statement that a_n and b_n are distributed normally.

In view of this wealth of information on the subject it may appear superfluous to say anything about it. However, for the sake of completeness, we shall outline the thoughts which lead to (1.7–1).

In line with our usual approach to the shot effect, we suppose that exactly K electrons arrive during the interval $(0, T)$, so that the noise current for the interval is

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.7–2}$$

The coefficients in the Fourier series expansion of $I_K(t)$ over the interval $(0, T)$ are a_{nK} and b_{nK} where

$$\begin{aligned} a_{nK} - ib_{nK} &= \frac{2}{T} \sum_{k=1}^{K} \int_0^T F(t - t_k) \exp\left[-i\,\frac{2\pi nt}{T}\right] dt \\ &\approx \frac{2}{T} \sum_{k=1}^{K} \int_{-\infty}^{+\infty} F(t) \exp\left[-i\,\frac{2\pi n}{T}\,(t + t_k)\right] dt \\ &= R_n e^{-i\varphi_n} \sum_{k=1}^{K} e^{-in\theta_k} \end{aligned} \tag{1.7–3}$$

In this expression

$$\begin{aligned} \theta_k &= \frac{2\pi t_k}{T} \\ R_n e^{-i\varphi_n} &= C_n - iS_n = \frac{2}{T} \int_{-\infty}^{+\infty} F(t) e^{-i2\pi nt/T}\, dt \end{aligned} \tag{1.7–4}$$

In the earlier sections the arrival times $t_1, t_2, \cdots t_K$ were regarded as K independent random variable each distributed uniformly over the interval (O, T). Hence the θ_k's may be regarded as random variables distributed uniformly over the interval 0 to 2π.

Incidentally, it will be noted that in (1.7–3) there occurs the sum of K randomly oriented unit vectors. When K becomes very large, as it does

[8] *Ann. d. Physik*, 57 (1918) pp. 541–567.

[9] Unpublished Memorandum, "Fluctuations in Vacuum Tube Noise and the Like," March 17, 1932.

[10] Investigation of Hidden Periodicities, Terrestrial Magnetism, 3 (1898), pp. 13–41. See especially propositions 1 and 2 on page 26 of Schuster's paper.

when $\nu \to \infty$, it is known that the real and imaginary parts of this sum are random variables, which tend to become independent and normally distributed about zero. This suggests the manner in which the normal distribution of the coefficients arises. Averaging over the θ_k's in (1.7–3) gives when $n > 0$

$$\bar{a}_{nK} = \bar{b}_{nK} = 0 \tag{1.7–5}$$

Some further algebra gives

$$\begin{gathered} \overline{a_{nK}^2} = \overline{b_{nK}^2} = \frac{K}{2} R_n^2 \\ \overline{a_{nK} b_{nK}} = \overline{a_{nK} a_{mK}} = \overline{b_{nK} b_{mK}} = 0 \end{gathered} \tag{1.7–6}$$

where $n \neq m$ and $n, m > 0$.

So far, we have been considering the case of exactly K arrivals in our interval of length T. Now we pass to the general case of any number of arrivals by making use of formulas analogous to

$$\overline{a_n^2} = \sum_{K=0}^{\infty} p(K) \overline{a_{nK}^2} \tag{1.7–7}$$

as has been done in section 1.3. Thus, for $n > 0$,

$$\begin{gathered} \bar{a}_n = \bar{b}_n = 0 \\ \overline{a_n^2} = \overline{b_n^2} = \frac{\nu T}{2} R_n^2 = \sigma_n^2 \\ \overline{a_n b_n} = \overline{a_n a_m} = \overline{b_n b_m} = 0, \qquad n \neq m \end{gathered} \tag{1.7–8}$$

In the second line we have used σ_n to denote the standard deviation of a_n and b_n. We may put the expression for σ_n^2 in a somewbat different form by writing

$$f_n = \frac{n}{T} = n\Delta f, \qquad \Delta f = \frac{1}{T} \tag{1.7–9}$$

where f_n is the frequency of the nth component. Using (1.7-4),

$$\sigma_n^2 = 2\nu\Delta f \left| \int_{-\infty}^{+\infty} F(t) e^{-i2\pi f_n t} \, dt \right|^2 \tag{1.7–10}$$

Thus, σ_n^2 is proportional to ν/T.

Tbe probability density function $P(a_1, \cdots a_N, b_1, \cdots b_N)$ for the $2N$ coefficients, $a_1, \cdots a_N, b_1, \cdots b_N$ may be derived in much the same fashion as was the probability density of the noise current in section 1.4. Here N

is arbitrary but fixed. The expression analogous to (1.4–5) is tbe $2N$ fold integral

$$P(a_1, \cdots, b_N) = (2\pi)^{-2N} \int_{-\infty}^{+\infty} du_1 \cdots \int_{-\infty}^{+\infty} dv_N \qquad (1.7\text{–}11)$$

$$\exp\left[-i(a_1 u_1 + \cdots + b_N v_N) - \nu T + \nu T E\right]$$

where

$$E = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i \sum_{n=1}^{N} (u_n C_n + v_n S_n) \cos n\theta + (v_n C_n - u_n S_n) \sin n\theta \right] \qquad (1.7\text{–}12)$$

in which $C_n - iS_n$ is defined as the Fourier transform (1.7–4) of $F(t)$.

The next step is to show that (1.7–11) approaches a normal law in $2N$ dimensions as $\nu \to \infty$. This appears to be quite involved. It will be noted that the integrand in the integral defining E is composed of N factors of the form

$$\exp\left[i\rho_n \cos(n\theta - \psi_n)\right]$$
$$= J_0(\rho_n) + 2i \cos(n\theta - \psi_n) J_1(\rho_n) - 2\cos(2n\theta - 2\psi_n) J_2(\rho_n) + \cdots$$

wbere

$$\rho_n^2 = (u_n^2 + v_n^2)(C_n^2 + S_n^2) = \frac{2}{\nu T}\sigma_n^2(u_n^2 + v_n^2).$$

As ν becomes large, it turns out that the integral (1.7–11) for the probability density obtains most of its contributions from small values of u and v. By substituting the product of the Bessel function series in the integral for E and integrating we find

$$E = \prod_{n=1}^{N} J_0(\rho_n) + A + B + C$$

where A is the sum of products such as

$$-2i \cos(\psi_{k+l} - \psi_k - \psi_l) J_1(\rho_k) J_1(\rho_l) J_1(\rho_{k+l}) \text{ times } N - 3\ J_0\text{'s}$$

in which $0 < k \leq l$ and $2 \leq k + l \leq N$. Similarly B is the sum of products of the form

$$-2i \cos(\psi_{2k} - 2\psi_k) J_1(\rho_{2k}) J_2(\rho_k) \text{ times } N - 2\ J_0\text{'s}$$

C consists of terms which give fourth and higher powers in u and v. There are roughly $N^2/4$ terms of form A and $N/2$ terms of form B.

Expanding the Bessel functions, neglecting all powers above the third and proceeding as in section 1.4, will give us the normal distribution plus the first correction term. It is rather a messy affair. An idea of how it looks may be obtained by taking the special case in which $F(t)$ is an even function cf t and neglecting terms of type B. Then

$$P(a_1, \cdots a_N, b_1, \cdots b_N) = (1 + \eta) \prod_{n=1}^{N} \frac{e^{-(x_n^2+y_n^2)/2}}{2\pi\sigma_n^2} \qquad (1.7\text{–}12)$$

where

$$x_n = \frac{a_n}{\sigma_n}, \qquad y_n = \frac{b_n}{\sigma_n}$$

$$\eta = (2\nu T)^{-1/2} \sum_{k,\ell} [x_{k+\ell}(x_k x_\ell - y_k y_\ell) + 2\, y_{k+\ell} y_k y_\ell] \qquad (1.7\text{–}13)$$

and the summation extends over $2 \leq k + l \leq N$ with $k \leq l$.

It is seen that if T and N are held constant, the correction term η approaches zero as ν becomes very large. A very rough idea of the magnitude of η may be obtained by assuming that unity is a representative value of the x's and y's. Further assuming that there are N^2 terms in the summation each one of which may be positive or negative suggests that magnitude of the sum is of the order of N. Hence we might expect to find that η is of the order of $N(2\nu T)^{-1/2}$.

PART II

POWER SPECTRA AND CORRELATION FUNCTIONS

2.0 Introduction

A theory for analyzing functions of time, t, which do not die down and which remain finite as t approaches infinity has gradually been developed over the last sixty years. A few words of its history together with an extensive bibliography are given by N. Wiener in his paper on "Generalized Harmonic Analysis".[11] G. Gouy, Lord Rayleigh and A. Schuster were led to study this problem in their investigations of such things as white light and noise. Schuster[12] invented the "periodogram" method of analysis which has as its object the discovery of any periodicities hidden in a continuous curve representing meteorological or economic data.

[11] *Acta Math.*, Vol. 55, pp. 117–258 (1930). See also "Harmonic Analysis of Irregular Motion," *Jour. Math. and Phys.* 5 (1926) pp. 99–189.

[12] The periodogram was first introduced by Schuster in reference 10 cited in Section 1.7. He later modified its definition in the Trans. Camb. Phil. Soc. 18 (1900), pp. 107–135, and still later redefined it in "The Periodogram and its Optical Analogy," Proc. Roy. Soc., London, Ser. A, 77 (1906) pp. 136–140. In its final form the periodogram is equivalent to $\frac{1}{4}w(f)$, where $w(f)$ is the power spectrum defined in Section 2.1, plotted as a function of the period $T = (2\pi f)^{-1}$.

The correlation function, which turns out to be a very useful tool, apparently was introduced by G. I. Taylor.[13] Recently it has been used by quite a few writers[14] in the mathematical theory of turbulence.

In section 2.1 the power spectrum and correlation function of a specific function, such as one given by a curve extending to $t = \infty$, are defined by equations (2.1–3) and (2.1–4) respectively. That they are related by the Fourier inversion formulae (2.1–5) and (2.1–6) is merely stated; the discussion of the method of proof being delayed until sections 2.3 and 2.4. In section 2.3 a discussion based on Fourier series is given and in section 2.4 a parallel treatment starting with Parseval's integral theorem is set forth. The results as given in section 2.1 have to be supplemented when the function being analyzed contains a d.c. or periodic components. This is taken up in section 2.2.

The first four sections deal with the analysis of a specific function of t. However, most of the applications are made to functions which behave as though they are more or less random in character. In the mathematical analysis this randomness is introduced by assuming the function of t to be also a function of suitable parameters, and then letting these parameters be random variables. This question is taken up in section 2.5. In section 2.6 the results of 2.5 are applied to determine the average power spectrum and the average correlation function of the shot effect current. The same thing is done in 2.7 for a flat top wave, the tops (and bottoms) being of random length. The case in which the intervals are of equal length but the sign of the wave is random is also discussed in 2.7. The representation of the noise current as a trigonometrical series with random variable coefficients is taken up in 2.8. The last two sections 2.9 and 2.10 are devoted to probability theory. The normal law and the central limit theorem, respectively, are discussed.

2.1 Some Results of Generalized Harmonic Analysis

We shall first state the results which we need, and then show that they are plausible by methods which are heuristic rather than rigorous. Suppose that $I(t)$ is one of the functions mentioned above. We may think of it as being specified by a curve extending from $t = -\infty$ to $t = \infty$. $I(t)$ may be regarded as composed of a great number of sinusoidal components whose frequencies range from 0 to $+\infty$. It does not necessarily have to be a noise current, but if we think of it as such, then, in flowing through a resistance of one ohm it will dissipate a certain average amount of power, say p watts.

[13] Diffusion by Continuous Movements, *Proc. Lond. Math. Soc.*, Ser. 2, 20, pp. 196–212 (1920).

[14] See the text "Modern Developments in Fluid Dynamics" edited by S. Goldstein, Oxford (1938).

That portion of ρ arising from the components having frequencies between f and $f + df$ will be denoted by $w(f)df$, and consequently

$$\rho = \int_0^\infty w(f)df \tag{2.1-1}$$

Since $w(f)$ is the spectrum of the average power we shall call it the "power spectrum" of $I(t)$. It has the dimensions of energy and on this account is frequently called the "energy-frequency spectrum" of $I(t)$. A mathematical formulation of this discussion leads to a clear cut definition of $w(f)$.

Let $\Phi(t)$ be a function of t, which is zero outside the interval $0 \leq t \leq T$ and is equal to $I(t)$ inside the interval. Its spectrum $S(f)$ is given by

$$S(f) = \int_0^T I(t)e^{-2\pi i f t}\, dt \tag{2.1-2}$$

The spectrum of the power, $w(f)$, is defined as

$$w(f) = \underset{T\to\infty}{\text{Limit}} \frac{2|S(f)|^2}{T} \tag{2.1-3}$$

where we consider only values of $f > 0$ and assume that this limit exists. This is substantially the definition of $w(f)$ given by J. R. Carson[15] and is useful when $I(t)$ has no periodic terms and no d.c. component. In the latter case (2.1–3) must either be supplemented by additional definitions or else a somewhat different method of approach used. These questions will be discussed in section 2.2.

The correlation function $\psi(\tau)$ of $I(t)$ is defined by the limit

$$\psi(\tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T I(t)I(t + \tau)\, dt \tag{2.1-4}$$

which is assumed to exist. $\psi(\tau)$ is closely related to the correlation coefficients used in statistical theory to measure the correlation of two random variables. In the present case the value of $I(t)$ at time t is one variable and its value at a different time $t + \tau$ is the other variable.

The spectrum of the power $w(f)$ and the correlation function $\psi(\tau)$ are related by the equations

$$w(f) = 4 \int_0^\infty \psi(\tau) \cos 2\pi f\tau\, d\tau \tag{2.1-5}$$

$$\psi(\tau) = \int_0^\infty w(f) \cos 2\pi f\tau\, df \tag{2.1-6}$$

[15] "The Statistical Energy-Frequency Spectrum of Random Disturbances," *B.S.T.J.*, Vol. 10, pp. 374–381 (1931).

It is seen that $\psi(\tau)$ is an even function of τ and that

$$\psi(0) = \rho \tag{2.1-7}$$

When either $\psi(\tau)$ or $w(f)$ is known the other may be obtained provided the corresponding integral converges.

2.2 Power Spectrum for D.C. and Periodic Components

As mentioned in section 2.1, when $I(t)$ has a d.c. or a periodic component the limit in the definition (2.1–3) for $w(f)$ does not exist for f equal to zero or to the frequency of the periodic component. Perhaps the most satisfactory method of overcoming this difficulty, from the mathematical point of view, is to deal with the integral of the power spectrum.[16]

$$\int_0^f w(g)\, dg \tag{2.2-1}$$

instead of with $w(f)$ itself.

The definition (2.1–4) for $\psi(\tau)$ still holds. If, for example,

$$I(t) = A + C \cos (2\pi f_0 t - \varphi) \tag{2.2-2}$$

$\psi(\tau)$ as given by (2.1–4) is

$$\psi(\tau) = A^2 + \frac{C^2}{2} \cos 2\pi f_0 \tau \tag{2.2-3}$$

The inversion formulas (2.1–5) and (2.1–6) give

$$\begin{aligned} \int_0^f w(g)\, dg &= \frac{2}{\pi} \int_0^\infty \psi(\tau) \frac{\sin 2\pi f \tau}{\tau}\, d\tau \\ \psi(\tau) &= \int_0^\infty \cos 2\pi f \tau\, d\left[\int_0^f w(g)\, dg\right] \end{aligned} \tag{2.2-4}$$

[16] This is done by Wiener,[11] loc. cit., and by G. W. Kenrick, "The Analysis of Irregular Motions with Applications to the Energy Frequency Spectrum of Static and of Telegraph Signals," *Phil. Mag.*, Ser. 7, Vol. 7, pp. 176–196 (Jan. 1929). Kenrick appears to be one of the first to apply, to noise problems, the correlation function method of computing the power spectrum (one of his problems is discussed in Sec. 2.7). He bases his work on results due to Wiener. Khintchine, in "Korrelationstheorie der stationären stochastischen Prozesse," *Math. Annalen*, 109 (1934), pp. 604–615, proves the following theorem: A necessary and sufficient condition that a function $R(t)$ may be the correlation function of a continuous, stationary, stochastic process is that $R(t)$ may be expressed as

$$R(t) = \int_{-\infty}^{+\infty} \cos tx\, dF(x)$$

where $F(x)$ is a certain distribution function. This expression for $R(t)$ is essentially the second of equations (2.2–4). Khintchine's work has been extended by H. Cramér, "On the theory of stationary random processes," *Ann. of Math.*, Ser. 2, Vol. 41 (1940), pp. 215–230. However, Khintchine and Cramér appear to be interested primarily in questions of existence, representation, etc., and do not stress the concept of the power spectrum.

where the last integral is to be regarded as a Stieltjes' integral. When the expression (2.2–3) for $\psi(\tau)$ is placed in the first formula of (2.2–4) we get

$$\int_0^f w(g)\,dg = \begin{cases} A^2 & \text{when} \quad 0 < f < f_0 \\ A^2 + \dfrac{C^2}{2}, & \text{“} \quad f > f_0 \end{cases} \tag{2.2–5}$$

When this expression is used in the second formula of (2.2–4), the increments of the differential are seen to be A^2 at $f = 0$ and $C^2/2$ at $f = f_0$. The resulting expression for $\psi(\tau)$ agrees with the original one.

Here we desire to use a less rigorous, but more convenient, method of dealing with periodic components. By examining the integral of $w(f)$ as given by (2.2–5) we are led to write

$$w(f) = 2A^2\delta(f) + \frac{C^2}{2}\delta(f - f_0) \tag{2.2–6}$$

where $\delta(x)$ is an even unit impulse function so that if $\epsilon > 0$

$$\int_0^\epsilon \delta(x)\,dx = \frac{1}{2}\int_{-\epsilon}^\epsilon \delta(x)\,dx = \tfrac{1}{2} \tag{2.2–7}$$

and $\delta(x) = 0$ except at $x = 0$, where it is infinite. This enables us to use the simpler inversion formulas of section 2.1. The second of these, (2.1-6), is immediately seen to give the correct expression for $\psi(\tau)$. The first one, (2.1–5), gives the correct expression for $w(f)$ provided we interpret the integrals as follows:

$$\begin{aligned} &\int_0^\infty \cos 2\pi f\tau\,d\tau = \tfrac{1}{2}\delta(f) \\ &\int_0^\infty \cos 2\pi f_0\tau \cos 2\pi f\tau\,d\tau = \tfrac{1}{4}\delta(f - f_0) \end{aligned} \tag{2.2–8}$$

It is not hard to show that these are in agreement with the fundamental interpretation

$$\int_{-\infty}^{+\infty} e^{-i2\pi ft}\,dt = \int_{-\infty}^{+\infty} e^{i2\pi ft}\,dt = \delta(f) \tag{2.2–9}$$

which in its turn follows from a formal application of the Fourier integral formula and

$$\int_{-\infty}^{+\infty} \delta(f)e^{i2\pi ft}\,df = \int_{-\infty}^{+\infty} \delta(f)e^{-i2\pi ft}\,df = 1 \tag{2.2–10}$$

We must remember that $f_0 > 0$ and $f \geq 0$ in (2.2–8) so that $\delta(f + f_0) = 0$ for $f \geq 0$.

The definition (2.1–3) for $w(f)$ gives the continuous part of the power spectrum. In order to get the part due to the d.c. and periodic components, which is exemplified by the expression (2.2–6) for $w(f)$ involving the δ functions, we must supplement (2.1–3) by adding terms of the type

$$2A^2\,\delta(f) + \frac{C^2}{2}\,\delta(f-f_0) = \left[\operatorname*{Limit}_{T\to\infty} \frac{2|S(0)|^2}{T^2}\right]\delta(f) + \left[\operatorname*{Limit}_{T\to\infty} \frac{2|S(f_0)|^2}{T^2}\right]\delta(f-f_0) \tag{2.2–11}$$

The correctness of this expression may be verified by calculating $S(f)$ for the $I(t)$ of this section given by (2.2–2), and actually carrying out the limiting process.

2.3 Discussion of Results of Section One—Fourier Series

The fact that the spectrum of power $w(f)$ and the correlation function $\psi(\tau)$ are related by Fourier inversion formulas is closely connected with Parseval's theorems for Fourier series and integrals. In this section we shall not use Parseval's theorems explicitly. We start with Fourier's series and use the concept of each component dissipating its share of energy independently of the behavior of the other components.

Let that portion of $I(t)$ which lies in the interval $0 \le t < T$ be expanded in the Fourier series

$$I(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nt}{T} + b_n \sin\frac{2\pi nt}{T}\right) \tag{2.3–1}$$

where

$$a_n = \frac{2}{T}\int_0^T I(t)\cos\frac{2\pi nt}{T}\,dt$$
$$b_n = \frac{2}{T}\int_0^T I(t)\sin\frac{2\pi nt}{T}\,dt \tag{2.3–2}$$

Then for the interval $-\tau \le t < T - \tau$,

$$I(t+\tau) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi n(t+\tau)}{T} + b_n \sin\frac{2\pi n\,(t+\tau)}{T}\right) \tag{2.3–3}$$

Multiplying the series for $I(t)$ and $I(t+\tau)$ together and integrating with respect to t gives, after some reduction,

$$\frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{1}{2}(a_n^2+b_n^2)\cos\frac{2\pi n}{T}\tau + O\left(\frac{\tau I^2}{T}\right) \tag{2.3–4}$$

where the last term represents correction terms which must be added because the series (2.3–3) does not represent $I(t + \tau)$ in the interval $(T - \tau, T)$ when $\tau > 0$, or in the interval $(0, -\tau)$ if $\tau < 0$.

If $I(t)$ were a current and if it were to flow through one ohm for the interval $(0, T)$, each component would dissipate a certain average amount of power. The average power dissipated by the component of frequency $f_n = n/T$ cycles per second would be, from the Fourier series and elementary principles,

$$\begin{aligned} &\frac{1}{2}(a_n^2 + b_n^2) \text{ watts}, && n \neq 0 \\ &\frac{a_0^2}{4} \text{ watts}, && n = 0 \end{aligned} \tag{2.3–5}$$

The band width associated with the nth component is the difference in frequency between the $n + 1$ th and nth components:

$$f_{n+1} - f_n = \frac{n+1}{T} - \frac{n}{T} = \frac{1}{T}\, cps$$

Hence if the average power in the band $f, f + df$ is defined as $w(f)df$, the average power in the band $f_{n+1} - f_n$ is

$$w(f_n)(f_{n+1} - f_n) = w\left(\frac{n}{T}\right)\frac{1}{T}$$

and, from (2.3–5), this is given by

$$\begin{aligned} &w\left(\frac{n}{T}\right)\frac{1}{T} = \frac{1}{2}(a_n^2 + b_n^2), && n \neq 0 \\ &w(0)\frac{1}{T} = \frac{a_0^2}{4}, && n = 0 \end{aligned} \tag{2.3–6}$$

When the coefficients in (2.3–4) are replaced by their expressions in terms of $w(f)$ we get

$$\begin{aligned} \frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt + O\left(\frac{\tau I^2}{T}\right) &= \frac{1}{T}\sum_{n=0}^{\infty} w\left(\frac{n}{T}\right)\cos\frac{2\pi n\tau}{T} \\ &= \int_0^{\infty} w\left(\frac{n}{T}\right)\cos\frac{2\pi n\tau}{T}\frac{dn}{T} \\ &= \int_0^{\infty} w(f)\cos 2\pi f\tau\, df \end{aligned} \tag{2.3–7}$$

where we have assumed T so large and $w(f)$ of such a nature that the summation may be replaced by integration.

If I remains finite, then as $T \to \infty$ with τ held fixed, the correction term on the left becomes negligibly small and we have, upon using the definitions (2.1–4) for the correlation function $\psi(\tau)$, the second of the fundamental inversion formulas (2.1–6). The first inversion formula may be obtained from this at once by using Fourier's double integral for $w(f)$.

Incidentally, the relation (2.3–6) between $w(f)$ and the coefficients a_n and b_n is in agreement with the definition (2.1–3) for $w(f)$ as a limit involving $| S(f) |^2$. From the expressions (2.3–2) for a_n and b_n, the spectrum $S(f_n)$ given by (2.1–2) is

$$S(f_n) = \frac{T}{2} (a_n - ib_n)$$

Then, from (2.1–3) $w(f_n)$ is given by the limit, as $T \to \infty$, of

$$\begin{aligned}\frac{2}{T} | S(f_n) |^2 &= \frac{2}{T} \cdot \frac{T^2}{4} (a_n^2 + b_n^2) \\ &= \frac{T}{2} (a_n^2 + b_n^2)\end{aligned}$$

and this is the expression for $w\left(\frac{n}{T}\right)$ given by (2.3–6).

2.4 Discussion of Results of Section One—Parseval's Theorem

The use of Parseval's theorem[17] enables us to derive the results of section 2.1 more directly than the method of the preceding section. This theorem states that

$$\int_{-\infty}^{+\infty} F_1(f)F_2(f)\, df = \int_{-\infty}^{+\infty} G_1(t)G_2(-t)\, dt \tag{2.4–1}$$

where F_1, G_1 and F_2, G_2 are Fourier mates related by

$$\begin{aligned}F(f) &= \int_{-\infty}^{+\infty} G(t)e^{-i2\pi ft}\, dt \\ G(t) &= \int_{-\infty}^{+\infty} F(f)e^{i2\pi ft}\, df\end{aligned} \tag{2.4–2}$$

It may be proved in a formal manner by replacing the F_1 on the left of (2.4–1) by its expression as an integral involving $G_1(t)$. Interchanging the

[17] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford (1937).

order of integration and using the second of (2.4–2) to replace F_2 by G_2 gives the right hand side.

We now set $G_1(t)$ and $G_2(t)$ equal to zero except for intervals of length T. These intervals and the corresponding values of G_1 and G_2 are

$$G_1(t) = I(t), \qquad 0 < t < T \tag{2.4–3}$$
$$G_2(t) = I(-t + \tau), \quad \tau - T < t < \tau$$

From (2.4–3) it follows that $F_1(f)$ is the spectrum $S(f)$ of $I(t)$ given hy equation (2.1–2). Since $I(t)$ is real it follows from the first of equations (2.4–2) that

$$S(-f) = S^*(f), \tag{2.4–4}$$

where the star denotes conjugate complex, and hence that $|S(f)|^2$ is an even function of f.

The first of equations (2.4–2) also gives

$$\begin{aligned} F_2(f) &= \int_{\tau-T}^{\tau} I(-t+\tau)e^{-i2\pi f t}\,dt \\ &= \int_0^T I(t)e^{i2\pi f(t-\tau)}\,dt \\ &= S^*(f)e^{-i2\pi f\tau} \end{aligned} \tag{2.4–5}$$

When these G's and F's are placed in (2.4–1) we obtain

$$\int_{-\infty}^{+\infty} |S(f)|^2 e^{-2\pi f\tau}\,df = \int_0^{T-\tau} I(t)I(t+\tau)\,dt \tag{2.4–6}$$

where we have made use of the fact that $G_2(-t)$ is zero except in the interval $-\tau < t < T - \tau$ and have assumed $\tau > 0$. If $\tau < 0$ the limits of integration on the right would be $-\tau$ and T.

Since $|S(f)|^2$ is an even function of f we may write (2.4–6) as

$$\frac{1}{T}\int_0^T I(t)I(t+\tau)\,dt + O\left(\frac{\tau I^2}{T}\right) = \int_0^\infty \frac{2|S(f)|^2}{T}\cos 2\pi f\tau\,df \tag{2.4–7}$$

If we now define the correlation function $\psi(\tau)$ as the limit, as $T \to \infty$, of the left hand side and define $w(f)$ as the function

$$w(f) = \operatorname*{Limit}_{T\to\infty} \frac{2|S(f)|^2}{T}, \qquad f > 0 \tag{2.1–3}$$

we obtain the second, (2.1–6), of the fundamental inversion formulas. As before, the first may be obtained from Fourier's integral theorem.

In order to obtain the interpretation of $w(f)df$ as the average power dissipated in one ohm by those components of $I(t)$ which lie in the band f, $f + df$, we set $\tau = 0$ in (2.4–7):

$$\underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T I^2(t)\, dt = \int_0^\infty w(f)\, df \tag{2.4–8}$$

The expression on the left is certainly the total average power which would be dissipated in one ohm and the right hand side represents a summation over all frequencies extending from 0 to ∞. It is natural therefore to interpret $w(f)df$ as the power due to the components in $f, f + df$.

The preceding sections have dealt with the power spectrum $w(f)$ and correlation function $\psi(\tau)$ of a very general type of function. It will be noted that a knowledge of $w(f)$ does not enable us to determine the original function. In obtaining $w(f)$, as may be seen from the definition (2.1–3) or from (2.3–6), the information carried by the phase angles of the various components of $I(t)$ has been dropped out. In fact, as we may see from the Fourier series representation (2.3–1) of $I(t)$ and from (2.3–6), it is possible to obtain an infinite number of different functions all of which have the same $w(f)$, and hence the same $\psi(\tau)$. All we have to do is to assign different sets of values to the phase angles of the various components, thereby keeping $a_n^2 + b_n^2$ constant.

2.5 Harmonic Analysis for Random Functions

In many applications of the theory discussed in the foregoing sections $I(t)$ is a function of t which has a certain amount of randomness associated with it. For example $I(t)$ may be a curve representing the price of wheat over a long period of years, a component of air velocity behind a grid placed in a wind tunnel, or, of primary interest here, a noise current.

In some mathematical work this randomness is introduced by considering $I(t)$ to involve a number of parameters, and then taking the parameters to be random variables. Thus, in the shot effect the arrival times $t_1, t_2, \cdots t_K$ of the electrons were taken to be the parameters and each was assumed to he uniformly distributed over an interval $(0, T)$.

For any particular set of values of the parameters, $I(t)$ has a definite power spectrum $w(f)$ and correlation function $\psi(\tau)$. However, now the principal interest is not in these particular functions, but in functions which give the average values of $w(f)$ and $\psi(\tau)$ for fixed f and τ. These functions are obtained by averaging $w(f)$ and $\psi(\tau)$ over the ranges of the parameters, using, of course, the distribution functions of the parameters.

By averaging both sides of the appropriate equations in sections 2.1 and

2.2 it is seen that our fundamental inversion formulae (2.1–5) and (2.1–6) are unchanged. Thus,

$$\bar{w}(f) = 4 \int_0^\infty \bar{\psi}(\tau) \cos 2\pi f\tau \, d\tau \tag{2.5–1}$$

$$\bar{\psi}(\tau) = \int_0^\infty \bar{w}(f) \cos 2\pi f\tau \, df \tag{2.5–2}$$

where the bars indicate averages taken over the parameters with f or τ held constant.

The definitions of $\bar{w}$ and $\bar{\psi}$ appearing in these equations are likewise obtained from (2.1–3) and (2.1–4)

$$\bar{w}(f) = \underset{T\to\infty}{\text{Limit}} \frac{2\overline{|S(f)|^2}}{T} \tag{2.5–3}$$

and

$$\bar{\psi}(\tau) = \underset{T\to\infty}{\text{Limit}} \frac{1}{T} \int_0^T \overline{I(t)I(t+\tau)} \, dt \tag{2.5–4}$$

The values of t and τ are held fixed while averaging over the parameters. In (2.5–3) $S(f)$ is regarded as a function of the parameters obtained from $I(t)$ by

$$S(f) = \int_0^T I(t)e^{-2\pi i f t} \, dt \tag{2.1–2}$$

Similar expressions may be obtained for the average power spectrum for d.c. and periodic components. All we need to do is to average the expression (2.2–11)

Sometimes the average value of the product $I(t)I(t + \tau)$ in the definition (2.5–4) of $\bar{\psi}(\tau)$ is independent of the time T. This enables us to perform the integration at once and obtain

$$\bar{\psi}(\tau) = \overline{I(t)I(t+\tau)} \tag{2.5–5}$$

This introduces a considerable simplification and it appears that the simplest method of computing $\bar{w}(f)$ for an $I(t)$ of this sort is first to compute $\bar{\psi}(\tau)$, and then use the inversion formula (2.5–1).

2.6 First Example—The Shot Effect

We first compute the average on the right of (2.5–5). By using the method of averaging employed many times in part I, we have

$$\overline{I(t)I(t+\tau)} = \sum_{K=0}^{\infty} p(K) \, \overline{I_K(t)I_K(t+\tau)} \tag{2.6–1}$$

where $p(K)$ is the probability of exactly K electrons arriving in the interval $(0, T)$,

$$p(K) = \frac{(\nu T)^K}{K!} e^{-\nu T} \tag{1.1-3}$$

and

$$I_K(t) = \sum_{k=1}^{K} F(t - t_k) \tag{1.3-1}$$

Multiplying $I_K(t)$ and $I_K(t + \tau)$ together and averaging $t_1, t_2, \cdots t_K$ over their ranges gives

$$\overline{I_K(t)I_K(t + \tau)} = \sum_{k=1}^{K} \sum_{m=1}^{K} \int_0^T \frac{dt_1}{T} \cdots \int_0^T \frac{dt_K}{T} F(t - t_k)F(t + \tau - t_m)$$

This is similar to the expression for $\overline{I_K^2(t)}$ which was used in section 1.3 to prove Campbell's theorem and may be treated in much the same way. Thus, if t and $t + \tau$ lie between Δ and $T - \Delta$, the expression above becomes

$$\frac{K}{T} \int_{-\infty}^{+\infty} F(t)F(t + \tau)\, dt + \frac{K(K - 1)}{T^2} \left[\int_{-\infty}^{+\infty} F(t)\, dt\right]^2$$

When this is placed in (2.6–1) and the summation performed we obtain an expression independent of T. Consequently we may use (2.5–5) and get

$$\bar{\psi}(\tau) = \nu \int_{-\infty}^{+\infty} F(t)F(t + \tau)\, dt + \overline{I(t)}^2 \tag{2.6-2}$$

where we have used the expression for the average current

$$\overline{I(t)} = \nu \int_{-\infty}^{+\infty} F(t)\, dt \tag{1.3-4}$$

In order to compute $\bar{w}(f)$ from $\bar{\psi}(\tau)$ it is convenient to make use of the fact that $\psi(\tau)$ is always an even function of τ and hence (2.5–1) may also be written as

$$\bar{w}(f) = 2 \int_{-\infty}^{+\infty} \bar{\psi}(\tau) \cos 2\pi f\tau\, d\tau \tag{2.6-3}$$

Then

$$\bar{w}(f) = 2\nu \int_{-\infty}^{+\infty} dt\, F(t) \int_{-\infty}^{+\infty} d\tau\, F(t + \tau) \cos 2\pi f\tau$$

$$+ 2 \int_{-\infty}^{+\infty} \overline{I(t)}^2 \cos 2\pi f\tau\, d\tau$$

$$= 2\nu \text{ Real Part of } \int_{\infty}^{+\infty} dt\, F(t)e^{-2\pi ift} \int_{\infty}^{+\infty} dt'\, F(t')e^{2\pi ift'}$$

$$+ 2\overline{I(t)}^2 \int_{\infty}^{+\infty} e^{i2\pi f\tau}\, d\tau$$

$$= 2\nu\, |\, s(f)\,|^2 + 2\overline{I(t)}^2 \delta(f) \qquad (2.6\text{–}4)$$

In going from the first equation to the second we have written $t' = t + \tau$ and have considered $\cos 2\pi f\tau$ to be the real part of the corresponding exponential. In going from the second equation to the third we have set

$$s(f) = \int_{-\infty}^{+\infty} F(t)e^{-2\pi ift}\, dt \qquad (2.6\text{–}5)$$

and have used

$$\int_{\infty}^{+\infty} e^{i2\pi ft}\, dt = \delta(f) \qquad (2.2\text{–}9)$$

The term in $\bar{w}(f)$ involving $\delta(f)$ represents the average power which would be dissipated by the d.c. component of $I(t)$ in flowing through one ohm. It is in agreement with the concept that the average power in the band $0 \le f < \epsilon$, $\epsilon > 0$ but very small, is

$$\begin{aligned} \int_0^{\epsilon} \bar{w}(f)\, dt &= 2\overline{I(t)}^2 \int_0^{\epsilon} \delta(f)\, df \\ &= \overline{I(t)}^2 \end{aligned} \qquad (2.6\text{–}6)$$

The expression (2.6–4) for $\bar{w}(f)$ may also be obtained from the definition (2.5–3) for $\bar{w}(f)$plus the additional term due to the d.c. component obtained by averaging the expressions (2.2–11). We leave this as an exercise for the reader. He will find it interesting to study the steps in Carson's[15] paper leading up to equation (8). Carson's $R(\omega)$ is related to our $\bar{w}(f)$ by

$$\bar{w}(f) = 2\pi R(\omega)$$

and his $f(i\omega)$ is equal to our $s(f)$.

Integrating both sides of (2.6–4) with respect to f from 0 to ∞ and using

$$\overline{I^2} = \int_0^{\infty} \bar{w}(f)\, df$$

gives the result

$$\overline{I^2} - \bar{I}^2 = 2\nu \int_0^{+\infty} |\, s(f)\,|^2\, df \qquad (2.6\text{–}7)$$

[15] Loc. cit.

This may be obtained immediately from Campbell's theorem by applying Parseval's theorem.

As an example of the use of these formulas we derive the power spectrum of the voltage across a resistance R when a current consisting of a great number of very short pulses per second flows through R. Let $F(t - t_k)$ be the voltage produced by the pulse occurring at time t_k. Then

$$F(t) = R\varphi(t)$$

where $\varphi(t)$ is the current in the pulse. We confine our interest to relatively low frequencies such that we may make the approximation

$$s(f) = \int_{-\infty}^{+\infty} R\varphi(t)e^{-2\pi i f t}\, dt$$

$$\approx R\int_{-\infty}^{+\infty} \varphi(t)\, dt = Rq$$

where q is the charge carried through the resistance by one pulse. From (2.6–4) it follows that for these low frequencies the continuous portion of the power spectrum for the voltage is constant and equal to

$$\tilde{w}(f) = 2\nu R^2q^2 = 2\bar{I}R^2q \qquad (2.6\text{–}8)$$

where $\bar{I} = \nu q$ is the average current flowing through R. This result is often used in connection with the shot effect in diodes.

In the study of the shot effect it was assumed that the probability of an event (electron arriving at the anode) happening in dt was νdt where ν is the expected number of events per second. This probability is independent of the time t. Sometimes we wish to introduce dependency on time.[18] As an example, consider a long interval extending from 0 to T. Let the probability of an event happening in $t, t + dt$ be $\bar{K}p(t)dt$ where $\bar{K}$ is the average number of events during T and $p(t)$ is a given function of t such that

$$\int_0^T p(t)\, dt = 1$$

For the shot effect $p(t) = 1/T$.

What is the probability that exactly K events happen in T? As in the case of the shot effect, section 1.1, we may divide $(0, T)$ into N intervals each of length Δt so that $N\Delta t = T$. The probability of no event happening in the first Δt is

$$1 - \bar{K}p\left(\frac{\Delta t}{2}\right)\Delta t$$

[18] A careful discussion of this subject is given by Hurwitz and Kac in "Statistical Analysis of Certain Types of Random Functions." I understand that this paper will soon appear in the Annals of Math. Statistics.

The product of N such probabilities is, as $N \rightarrow \infty$, $\Delta t \rightarrow 0$;

$$\exp\left[-\bar{K}\int_0^T p(t)\,dt\right] = e^{-\bar{K}}$$

This is the probability that exactly 0 events happen in T. In the same way we are led to the expression

$$\frac{\bar{K}^K}{K!}e^{-\bar{K}} \tag{2.6–9}$$

for the probability that exactly K events bappen in T.

When we consider many intervals $(0, T)$ we obtain many values of K and also many values of I measured t seconds from the beginning of each interval. These values of I define the distribution of I at time t. By proceeding as in section 1.4 we find that the probability density of I is

$$P(I, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du \exp\left[-iuI + \bar{K}\int_0^T p(x)(e^{iuF(t-x)} - 1)\,dx\right]$$

The corresponding average and variance is

$$\bar{I} = \bar{K}\int_0^T p(x)F(t - x)\,dx$$

$$\overline{(I - \bar{I})^2} = \bar{K}\int_0^T p(x)F^2(t - x)\,dx \tag{2.6–10}$$

If $S(f)$ is given by (2.1–2) and $s(f)$ by (2.6–5) (assuming the duration of $F(t)$ short in comparison with T) the average value of $|S(f)|^2$ may be obtained by putting (1.3-1) in (2.1–2) to get

$$S_K(f) = s(f)\sum_1^K e^{-2\pi i f t_k}$$

Expressing $S_K(f)\,S_K^*(f)$, where the star denotes conjugate complex, as a double sum and averaging over the t_k's, using $p(t)$, and then averaging over the K's gives

$$\overline{|S(f)|^2} = \bar{K}|s(f)|^2\left[1 + \bar{K}\left|\int_0^T p(x)e^{-2\pi i f x}\,dx\right|^2\right] \tag{2.6–11}$$

This may be used to compute the power spectrum from (2.5–3) provided $p(x)$ is not periodic. If $p(x)$ is periodic then the method of section 2.2 should be used at the harmonic frequencies. If the fluctuations of $p(t)$ are slow in comparison with the fluctuations of $F(t)$ the second term within the brackets of (2.6–11) may generally be neglected since there are no values of

f which make both it and $s(f)$ large at the same time. On the other hand, if both $p(t)$ and $F(t)$ fluctuate at about the same rate this term must be considered.

2.7 Second Example—Random Telegraph Signal[16]

Let $I(t)$ be equal to either a or $-a$ so that it is of the form of a flat top wave. Let the intervals between changes of sign, i.e. the lengths of the tops and bottoms, be distributed exponentially. We are led to this distribution by assuming that, if on the average there are μ changes of sign per second, the probability of a change of sign in $t, t + dt$ is μdt and is independent of what happens outside the interval $t, t + dt$. From the same sort of reasoning as employed in section 1.1 for the sbot effect we see that the probability of obtaining exactly K changes of sign in tbe interval $(0, T)$ is

$$p(K) = \frac{(\mu T)^K}{K!} e^{-\mu T} \tag{2.7–1}$$

We consider the average value of the product $I(t)I(t + \tau)$. This product is a^2 if the two I's are of the same sign and is $-a^2$ if they are of opposite sign. In the first case there are an even number, including zero, of changes of sign in the interval $(t, t + \tau)$, and in the second case there are an odd number of changes of sign. Thus

$$\begin{aligned} &\text{Average value of } I(t)I(t+\tau) \qquad (2.7\text{–}2)\\ &= a^2 \times \text{probability of an even number of changes of sign in } t, t+\tau \\ &- a^2 \times \text{probability of an odd number of changes of sign in } t, t+\tau \end{aligned}$$

The length of the interval under consideration is $|t + \tau - t| = |\tau|$ seconds. Since, by assumption, the probability of a change of sign in an elementary interval of length Δt is independent of what happens outside that interval, it follows that the same is true of any interval irrespective of wben it starts. Hence the probabilities in (2.7–2) are independent of t and may be obtained from (2.7–1) by setting $T = |\tau|$. Then (2.7–2) becomes, assuming $\tau > 0$ for the moment,

$$\begin{aligned} \overline{I(t)I(t+\tau)} &= a^2[p(0) + p(2) + p(4) + \cdots] \\ &\qquad - a^2[p(1) + p(3) + p(5) + \cdots] \\ &= a^2 e^{-\mu\tau}\left[1 - \frac{\mu\tau}{1!} + \frac{(\mu\tau)^2}{2!} - \cdots\right] \\ &= a^2 e^{-2\mu\tau} \end{aligned} \tag{2.7–3}$$

[16] Kenrick, cited in Section 2.2.

From (2.5–5), this gives the correlation function for $I(t)$

$$\bar{\psi}(\tau) = a^2 e^{-2\mu|\tau|} \tag{2.7-4}$$

The corresponding power spectrum is, from (2.5–1),

$$\begin{aligned} \bar{w}(f) &= 4a^2 \int_0^\infty e^{-2\mu\tau} \cos 2\pi f\tau \, d\tau \\ &= \frac{2a^2 \mu}{\pi^2 f^2 + \mu^2} \end{aligned} \tag{2.7-5}$$

Correlation functions and power spectra of this type occur quite frequently. In particular, they are of use in the study of turbulence in hydrodynamics. We may also obtain them from our shot effect expressions if we disregard the d.c. component. All we have to do is to assume that the effect $F(t)$ of an electron arriving at the anode at time $t = 0$ is zero for $t < 0$, and that $F(t)$ decays exponentially with time after jumping to its maximum value at $t = 0$. This may be verified by substituting the value

$$F(t) = 2a \sqrt{\frac{\mu}{\nu}}\, e^{-2\mu t}, \qquad t > 0 \tag{2.7-6}$$

for $F(t)$ in the expressions (2.6–2) and (2.6–4) (after using 2.6–5) for the correlation function and energy spectrum of the shot effect.

The power spectrum of the current flowing through an inductance and a resistance in series in response to a very wide band thermal noise voltage is also of the form (2.7–5).

Incidentally, this gives us an example of two quite different $I(t)$'s, one a flat top wave and the other a shot effect current, which have the same correlation functions and power spectra, aside from the d.c. component.

There is another type of random telegraph signal which is interesting to analyze. The time scale is divided into intervals of equal length h. In an interval selected at random the value of $I(t)$ is independent of the values in the other intervals, and is equally likely to be $+a$ or $-a$. We could construct such a wave by flipping a penny. If heads turned up we would set $I(t) = a$ in $0 < t < h$. If tails were obtained we would set $I(t) = -a$ in this interval. Flipping again would give either $+a$ or $-a$ for the second interval $h < t < 2h$, and so on. This gives us one wave. A great many waves may be constructed in this way and we denote averages over these waves, with t held constant, by bars.

We ask for the average value of $I(t)I(t + \tau)$, assuming $\tau > 0$. First we note that if $\tau > h$ the currents correspond to different intervals for all values of t. Since the values in these intervals are independent we have

$$\overline{I(t)I(t+\tau)} = \overline{I(t)}\ \overline{I(t+\tau)} = 0$$

for all values of t when $\tau > h$.

To obtain the average when $\tau < h$ we consider t to lie in the first interval $0 < t < h$. Since all the intervals are the same from a statistical point of view we lose no generality in doing this. If $t + \tau < h$, i.e., $t < h - \tau$, both currents lie in the first interval and

$$\overline{I(t)I(t+\tau)} = a^2$$

If $t > h - \tau$ the current $I(t + \tau)$ corresponds to the second interval and hence the average value is zero.

We now return to (2.5–4). The integral there extends from 0 to T. When $\tau > h$, the integrand is zero and hence

$$\bar{\psi}(\tau) = 0, \qquad \tau > h \tag{2.7–7}$$

When $\tau < h$, our investigation of the interval $0 < t < h$ enables us to write down the portion of the integral extending from 0 to h:

$$\int_0^h I(t)I(t+\tau)\,dt = \int_0^{h-\tau} a^2\,dt + \int_{h-\tau}^h 0\,dt$$
$$= a^2(h - \tau)$$

Over the interval of integration $(0, T)$ we have T/h such intervals each contributing the same amount. Hence, from (2.5–4),

$$\begin{aligned}\bar{\psi}(\tau) &= \operatorname*{Limit}_{T\to\infty} \frac{a^2}{T}\cdot\frac{T}{h}(h - \tau)\\ &= a^2\left(1 - \frac{\tau}{h}\right), \qquad 0 \leq \tau < h\end{aligned} \tag{2.7–8}$$

The power spectrum of this type of telegraph wave is thus

$$\begin{aligned}\bar{w}(f) &= 4a^2\int_0^h \left(1 - \frac{\tau}{h}\right)\cos 2\pi f\tau\,d\tau\\ &= 2h\left(\frac{a \sin \pi fh}{\pi fh}\right)^2\end{aligned} \tag{2.7–9}$$

This is seen to have the same general behavior as $\bar{w}(f)$ for the first type of telegraph signal given by (2.7–5), when we relate the average number, μ, of changes of sign per second to the interval length h by $\mu h = 1$.

2.8 Representation of Noise Current

In section 1.8 the Fourier series representation of the shot effect current was discussed. This suggests the representation*

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \tag{2.8-1}$$

where

$$\omega_n = 2\pi f_n, \qquad f_n = n\Delta f \tag{2.8-2}$$

a_n and b_n are taken to be independent random variables which are distributed normally about zero with the standard deviation $\sqrt{w(f_n)\Delta f}$. $w(f)$ is the power spectrum of the noise current, i.e., $w(f)\, df$ is the average power which would be dissipated by those components of $I(t)$ which lie in the frequency range $f, f + df$ if they were to flow through a resistance of one ohm.

The expression for the standard deviation of a_n and b_n is obtained when we notice that Δf is the width of the frequency band associated with the nth component. Hence $w(f_n)\Delta f$ is the average energy which would be dissipated if the current

$$a_n \cos \omega_n t + b_n \sin \omega_n t$$

were to flow through a resistance of one ohm, this average being taken over all possible values of a_n and b_n. Thus

$$w(f_n)\Delta f = \overline{a_n^2} \cos^2 \omega_n t + \overline{2a_n b_n} \cos \omega_n t \sin \omega_n t + \overline{b_n^2} \sin^2 \omega_n t = \overline{a_n^2} = \overline{b_n^2} \tag{2.8-3}$$

The last two steps follow from the independence of a_n and b_n and the identity of their distributions. It will be observed that $w(f)$, as used with the representation (2.8-1), is the same sort of average as was denoted in section 2.5 by $\bar{w}(f)$. However, $w(f)$ is often given to us in order to specify the spectrum of a given noise.

For example, suppose we are interested in the output of a certain filter when a source of thermal noise is applied to the input. Let $A(f)$ be the absolute value of the ratio of the output current to the input current when a steady sinusoidal voltage of frequency f is applied to the input. Then

$$w(f) = cA^2(f) \tag{2.8-4}$$

* As mentioned in section 1.7 this sort of representation was used by Einstein and Hopf for radiation. Shottky (1918) used (2.8-1), apparently without explicitly taking the coefficients to be normally distributed. Nyquist (1932) derived the normal distribution from the shot effect.

If W is the average power dissipated in one ohm by $I(t)$,

$$\begin{aligned} W &= \operatorname*{Limit}_{T\to\infty} \frac{1}{T}\int_0^T I^2(t)\,dt = \int_0^\infty w(f)\,df \\ &= c\int_0^\infty A^2(f)\,df \end{aligned} \tag{2.8–5}$$

which is an equation to determine c when W and $A(f)$ are known.

In using the representation (2.8–1) to investigate the statistical properties of $I(t)$ we first find the corresponding statistical properties of the summation on the right when the a's and b's are regarded as random variables distributed as mentioned above and t is regarded as fixed. In general, the time t disappears in this procedure just as it did in (2.8–3). We then let $N \to \infty$ and $\Delta f \to 0$ so that the summations may be replaced by integrations. Finally, the frequency range is extended to cover all frequencies from 0 to ∞.

The usual way of looking at the representation (2.8–1) is to suppose that we have an oscillogram of $I(t)$ extending from $t = 0$ to $t = \infty$. This oscillogram may be cut up into strips of length T. A Fourier analysis of $I(t)$ for each strip will give a set of coefficients. These coefficients will vary from strip to strip. Our representation ($T\Delta f = 1$) assumes that this variation is governed by a normal distribution. Our process for finding statistical properties by regarding the a's and b's as random variables while t is kept fixed corresponds to examining the noise current at a great many instants. Corresponding to each strip there is an instant, and this instant occurs at t (this is the t in (2.8–1)) seconds from the beginning of the strip. This is somewhat like examining the noise current at a great number of instants selected at random.

Although (2.8–1) is the representation which is suggested by the shot effect and similar phenomena, it is not the only representation, nor is it always the most convenient. Another representation which leads to the same results when the limits are taken is[19]

$$I(t) = \sum_{n=1}^{N} c_n \cos(\omega_n t - \varphi_n) \tag{2.8–6}$$

where $\varphi_1, \varphi_2, \cdots \varphi_N$ are angles distributed at random over the range $(0, 2\pi)$ and

$$c_n = [2w(f_n)\Delta f]^{1/2}, \qquad \omega_n = 2\pi f_n, \qquad f_n = n\Delta f \tag{2.8-7}$$

[19] This representation has often been used by W. R. Bennett in unpublished memoranda written in the 1930's.

In this representation $I(t)$ is regarded as the sum of a number of sinusoidal components with fixed amplitudes but random phase angles.

That the two different representations (2.8–1) and (2.8–6) of $I(t)$ lead to the same statistical properties is a consequence of the fact that they are always used in such a way that the "central limit theorem*" may be used in both cases.

This theorem states that under certain general conditions, the distribution of the sum of N random vectors approaches a normal law (it may be normal in several dimensions**) as $N \to \infty$. In fact from this theorem it appears that a representation such as

$$I(t) = \sum_{n=1}^{N} (a_n \cos \omega_n t + b_n \sin \omega_n t) \tag{2.8–6}$$

where a_n and b_n are independent random variables which take only the values $\pm [w(f_n)\Delta f]^{1/2}$, the probability of each value being $\frac{1}{2}$, will lead in the limit to the same statistical properties of $I(t)$ as do (2.8–1) and (2.8–6).

2.9 The Normal Distribution in Several Variables[20]

Consider a random vector r in K dimensions. The distribution of this vector may be specified by stating the distribution of the K components, $x_1, x_2, \cdots x_K$, of r. r is said to be normally distributed when the probability density function of the x's is of the form

$$(2\pi)^{-K/2} \mid M \mid^{-1/2} \exp [-\tfrac{1}{2}x'M^{-1}x] \tag{2.9–1}$$

where the exponent is a quadratic form in the x's. The square matrix M is composed of the second moments of the x's.

$$M = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1K} \\ \cdot & & & \cdot \\ \mu_{1K} & & \cdots & \mu_{KK} \end{bmatrix} \tag{2.9–2}$$

where the second moments are defined by

$$\mu_{11} = \overline{x_1^2}, \qquad \mu_{12} = \overline{x_1 x_2}, \qquad \text{etc.} \tag{2.9–3}$$

$\mid M \mid$ represents the determinant of M and x' is the row matrix

$$x' = [x_1, x_2, \cdots x_K] \tag{2.9–4}$$

x is the column matrix obtained by transposing x'.

* See section 2.10.
** See section 2.9.
[20] H. Cramér, "Random Variables and Probability Distributions," Chap. X., Cambridge Tract No. 36 (1937).

The exponent in the expression (2.9–1) for the probability density may be written out by using

$$x'M^{-1}x = \sum_{r=1}^{K}\sum_{s=1}^{K}\frac{M_{rs}}{|M|}x_r x_s \tag{2.9–5}$$

where M_{rs} is the cofactor of μ_{rs} in M.

Sometimes there are linear relations between the x's so that the random vector r is restricted to a space of less than K dimensions. In this case the appropriate form for the density function may be obtained by considering a sequence of K-dimensional distributions which approach the one being investigated.

If r_1 and r_2 are two normally distributed random vectors their sum $r_1 + r_2$ is also normally distributed. It follows that the sum of any number of normally distributed random vectors is normally distributed.

The characteristic function of the normal distribution is

$$\text{ave. } e^{iz_1x_1+iz_2x_2+\cdots+iz_Kx_K} = \exp\left[-\frac{1}{2}\sum_{r=1}^{K}\sum_{s=1}^{K}\mu_{rs}z_r z_s\right] \tag{2.9–6}$$

2.10 Central Limit Theorem

The central limit theorem in probability states that the distribution of the sum of N independent random vectors $r_1 + r_2 + \cdots + r_N$ approaches a normal law as $N \to \infty$ when the distributions of $r_1, r_2, \cdots r_N$ satisfy certain general conditions.[7]

As an example we take the case in which $r_1, r_2, \cdots$ are two-dimensional vectors[21], the components of r_n being x_n and y_n. Without loss of generality we assume that

$$\bar{x}_n = 0, \qquad \bar{y}_n = 0.$$

The components of the resultant vector are

$$\begin{aligned} X &= x_1 + x_2 + \cdots + x_N \\ Y &= y_1 + y_2 + \cdots + y_N \end{aligned} \tag{2.10–1}$$

and, since $r_1, r_2, \cdots$ are independent vectors, the second moments of the resultant are

$$\begin{aligned} \mu_{11} &= \overline{X^2} = \overline{x_1^2} + \overline{x_2^2} + \cdots + \overline{x_N^2} \\ \mu_{22} &= \overline{Y^2} = \overline{y_1^2} + \overline{y_2^2} + \cdots + \overline{y_N^2} \\ \mu_{12} &= \overline{XY} = \overline{x_1y_1} + \overline{x_2y_2} + \cdots + \overline{x_Ny_N} \end{aligned} \tag{2.10–2}$$

[7] Incidentally, von Laue (see references in section 1.7) used this theorem in discussing the normal distribution of the coefficients in a Fourier series used to represent black-body radiation. He ascribed it to Markoff.

[21] This case is discussed by J. V. Uspensky, "Introduction to Mathematical Probability", McGraw-Hill (1937) Chap. XV.

Apparently there are several types of conditions wbicb are sufficient to ensure that the distribution of the resultant approaches a normal law. One sufficient condition is that[21]

$$\mu_{11}^{-3/2} \sum_{n=1}^{N} \overline{|x_n|^3} \to 0$$
$$\mu_{22}^{-3/2} \sum_{n=1}^{N} \overline{|y_n|^3} \to 0 \qquad (2.10\text{–}3)$$

The central limit theorem tells us that tbe distribution of the random vector (X, Y) approaches a normal law as $N \to \infty$. The second moments of tbis distribution are given by (2.10–2). When we know the second moments of a normal distribution we may write down the probability density function at once. Thus from section 2.9

$$M = \begin{bmatrix} \mu_{11} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{bmatrix}, \qquad M^{-1} = |M|^{-1} \begin{bmatrix} \mu_{22} & -\mu_{12} \\ -\mu_{12} & \mu_{11} \end{bmatrix}$$

$$|M| = \mu_{11}\mu_{22} - \mu_{12}^2$$

$$x' = [X, Y]$$

$$x'M^{-1}x = |M|^{-1}(\mu_{22}X^2 - 2\mu_{12}XY + \mu_{11}Y^2)$$

The probability density is therefore

$$\frac{(\mu_{11}\mu_{22} - \mu_{12}^2)^{-1/2}}{2\pi} \exp\left[\frac{-\mu_{22}X^2 - \mu_{11}Y^2 + 2\mu_{12}XY}{2(\mu_{11}\mu_{22} - \mu_{12}^2)}\right] \qquad (2.10\text{–}3)$$

Incidentally, the second moments are related to the standard deviations σ_1, σ_2 of X, Y and to the correlation coefficient τ of X and Y by

$$\mu_{11} = \sigma_1^2, \qquad \mu_{22} = \sigma_2^2, \qquad \mu_{12} = \tau\sigma_1\sigma_2 \qquad (2.10\text{–}4)$$

and the probability density takes the standard form

$$\frac{(1 - \tau^2)^{-1/2}}{2\pi\sigma_1\sigma_2} \exp\left[-\frac{1}{2(1 - \tau^2)}\left(\frac{X^2}{\sigma_1^2} - 2\tau\frac{XY}{\sigma_1\sigma_2} + \frac{Y^2}{\sigma_2^2}\right)\right] \qquad (2.10\text{–}5)$$

[21] This is used by Uspensky, loc. cit. Another condition analogous to the Lindeberg condition is given by Cramer,[20] loc. cit.

(To be concluded)